PROJECT REPORT

MSc

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Construction and Analysis of Multi-Index Financial Model

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Abstract

This paper presents an empirical study of the expansion of the single-index model into a multi-index model with a careful selection of indices. A total of 91 multi-index models were built and tested using advanced statistical methods in the range from 2015 to 2018. This narrowed down the selection to 14 quality multi-index models which needed to be tested for the balance between simplicity and goodness of fit. Using the Akaike Information Criterion it was found that a seven-factor model provided the best trade-off between the two criteria. Nevertheless, it is up to the reader to decide what is of more importance, the fit quality or simplicity. Using the chi-square test, for the case of maximising the model quality benchmarked against the single-index model, a multi-index model has seen a reduction in chi-square value from 6588 to 4678. This model consisted of the market factor, 10 and 30-Year U.S. Treasury Bond, Gold, Oil, CMA and RMW which were proven to provide the most explanation behind the stock price movements as confirmed by the cited literature. Furthermore, this model was tested for its predictive power by testing it in the year 2019 which led to the conclusion that the model was overfitted due to the over-leniency of the Akaike Information Criterion and when it comes to predictiveness, a three-factor model is recommended. The last objective was to test the models in the extreme market events, so the range of 2020-2021 was selected which had unusual market movements caused by the global pandemic. The results reveal that none of the multi-index models outperformed the single-index model during the extreme market events due to the high correlation of all stocks with the market.

DECLARATION STATEMENT

I certify that the work submitted is my own and that any material derived or quoted from the published or unpublished work of other persons has been duly acknowledged (ref. UPR AS/C/6.1, Appendix I, Section 2 – Section on cheating and plagiarism)

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1. Introduction

The ability to predict the stock market performance has been a great challenge in the field of financial mathematics and it has been tackled since the 1900s, when the first scholarly work on mathematical finance has been published by a French mathematician Louis Bachelier. (Bachelier, 1900). Since then, there has been significant development in the field and several approaches have been taken to solve this problem. A particularly interesting approach was developed in the 1963., by an American economist William Sharpe who was the first to introduce a simple asset pricing model know as the single-index model. (Sharpe, 1963). It is based on the assumption that there is only one macroeconomic factor which causes the systematic risk affecting all stock returns and that is the rate of return of the market index, such as the S&P500. Observing such a market index and comparing it to the rate of return of a set of securities, it can be seen that there is a great deal of correlation between them, which implies that the single-index model carries a valid assumption, but it is not entirely true. The security prices can not be solely explained by the rate of return of the market index, but there is a large number of economic factors which influence the security price movements, and this is underlying idea behind this report. Since the original introduction of the single-index model there has been a great development on the topic and the model has been expanded to the multi-index model capturing multiple economical influences on the price movements. Some of the fundamental multi-index models include the Fama and French 3- and 5-factor models and Chen, Roll and Ross 7-factor model. (Fama, French, 1993; Chen, Roll, Ross, 1986).

Since there is a large number of economical influences on the price movements and the construction of the multi-index model is regarded as an extremely complex process, not because of the complexity of the math underlying it, but because the process itself can be considered as a form of art. The selection process of the factors, the testing of the performance, simplicity of the model and goodness of fit are all characteristics which must be considered when constructing the multi-index model. Finding the perfect balance between them is near impossible, but the aim of this thesis is focused on building the multi-index model to outperform the single-index model and this will require the completion of the following objectives:

- Creation of the benchmark portfolio with a small sample of stocks that will be used for initial testing.
- Using the python programming language, create and run the single-index model.
- Use a statistical method to measure the performance of the model.

- Build upon the single-index model with a set of indices and compare the performance.
- Expand the original portfolio to a large portfolio of approximately 250 stocks, to avoid an oversized influence of one or two companies.
- Investigation of the influential economical factors and selection of 10 to 15 factors for multi-index model testing
- Construction of the first set of multi-index model and finding the best performing index from the selected factors
- Continue the process until the order of the best performing indices is found
- Using a selection process, investigate the best ratio between the goodness of fit and simplicity of the multi-index model
- Validate the performance of the models by testing them on different years
- Draw final conclusions

2. Introduction into stock market analysis

Observing the largest stock exchanges around the world such as New York Stock Exchange, NASDAQ, London Stock Exchange, Japan Exchange Group and others where stockbrokers and traders buy and sell securities such as shares of company stocks, bonds and many other financial instruments there are thousands of different stocks and bonds to choose from. This presents an overwhelming number of opportunities for an investor when creating a portfolio of assets. Assuming that the investor prefers more to less, the objective is to create a range of efficient portfolios that offer the investor different levels of risk and return. Investor can then select the portfolio that best aligns with their risk tolerance and investment goals. (Elton E., Brown M., 2015) Even though, a portfolio, is simply a list of assets which is not difficult to create, managing a portfolio in the most efficient way requires significant skill. The process of portfolio management can be broken down into three components:

- Security Analysis
- Portfolio Analysis
- Portfolio Selection

2.1 Security Analysis

The first category is the "Security Analysis" where the individual focuses on the probability distributions of returns from a broad range of stocks, bonds and other financial instruments. In addition to that, the security analysis requires the analyst to make careful forecasts. Now, the period of forecast into the future should not be on an intra-day basis such as predictions for the next hour or day, nor they should be predictions on a super long term basis such as predicting the price of a stock or a bond in the next 10-15 years into the future as the prediction prices tend to be highly unrealistic. The forecasts made by the individual are presented in the terms of period rate of return which is evaluated in the following way:

$$\begin{pmatrix} Single\ period \\ rate\ of\ return \end{pmatrix} = \frac{\begin{pmatrix} Change\ in\ price\ during \\ the\ holding\ period \end{pmatrix} + \begin{pmatrix} Cash\ dividents\ paid\ during \\ the\ same\ period \end{pmatrix}}{\begin{pmatrix} Purchase\ price\ at\ the\ begining \\ of\ the\ holding\ period \end{pmatrix}}$$

Equation 1 - Rate of return (discrete return) (Capinski & Zastawniak, 2011)

Or written in the mathematical format:

$$r_1 = \frac{(P_1 - P_0) + d_1}{P_0}$$

Equation 2 - Discrete return (Capinski & Zastawniak, 2011)

Where:

$r_1 - Single\ period\ rate\ of\ return\ (or\ discrete\ return)$

$P_1 - Price\ of\ the\ security\ at\ the\ end\ of\ the\ holding\ period$

$P_0 - Price\ of\ the\ security\ at\ the\ beginning\ of\ the\ holding\ period$

$d_1 - Cash\ dividends\ paid\ during\ the\ holding\ period$

In the preparation for the portfolio construction, the individual should have a selection of different stocks and bonds anticipated for use as potential parts of their portfolio. Then, he should construct a probability distribution for each of the security selected based on their historical data and then adjust it subjectively in order to include factors that were not present in the past. In addition to a probability distribution, there are many steps to be conducted such as calculating correlation coefficients (covariances) between all the securities, the expected returns, average returns and variances required to evaluate the risk of the investments. (Elton E., Brown M., 2015) Risk is represented as variances of returns which is a measure of the difference between the outcomes and the average asset return. The greater the variance, the higher the risk is that investors could lose money. The variance of returns is defined by a statistical formula:

$$\sigma^2 = \frac{1}{N}\sum_{i=1}^{N}(r_i - \bar{r})^2$$

Equation 3 - Variance of Returns (Capinski & Zastawniak, 2011)

Where:

$\sigma^2 - Variance\ of\ returns$

$N - Number\ of\ periods\ observed$

$r_i - Return\ of\ the\ security$

$\bar{r} - Expected\ return\ of\ the\ security$

2.2 Portfolio Analysis

The second category “Portfolio Analysis” dates back to Nobel laureate Harry Markowitz who first introduced a mathematical algorithm for portfolio analysis in the 1950s. (Elton E., Brown M., 2015) In his model, there were three required inputs to the model:

- The expected rate of return, E(r), for each of the securities on the list for potential portfolio candidates.
- The standard deviation of the returns, σ, for each security as well
- And the correlation coefficients, ρ, between all of the selected securities

Markovitz’ portfolio analysis takes the three inputs stated above and performs the analysis resulting in a series of investment portfolios with a certain expected return rate and provides why each of the portfolios was selected and rejected according to it. The analysis also provides the weights of each security in the portfolio in that solution. From the analysis solution, the most desirable portfolios have the maximum expected rate of return at any given level of expected risk or, the minimum expected risk at any given expected rate of return. These types of solutions are known as Markowitz efficient assets which are better known as efficient portfolios. As there is more than one efficient portfolio depending on the investors' desirable risk and expected return criteria, the set of efficient portfolios is called "Efficient Frontier" which is shown in Figure 1. From the efficient frontier, an investor can choose what is his appetite for the risk and the desirable rate of return that goes with it which brings him to the last category, the portfolio selection. (Elton E., Brown M., 2015) However, before going into the portfolio selection process, there are four behavioural assumptions in the portfolio theory that need to be mentioned here.

1. The investor visualizes all of the investment opportunities by plotting the probability distribution of returns that is measured over the same holding period
2. The risk of the investment is determined by the variability of returns (standard deviation)
3. The investor is basing his investment decisions solely on the expected return and risk statistics. Which means that whatever return the investor receives from his investment, can be explained by E(r) and σ.
4. Lastly, the investor always prefers higher returns to lower returns at any given level of risk. Likewise, the investor prefers lower risk to higher risk at any given rate of return.

The four behavioural assumptions are logical, realistic and they uphold throughout the portfolio theory. From Figure 1., point E and F represent efficient portfolios with maximum rate of return for a certain amount of risk. No investor will for the rate of return of point E choose any of the portfolios to the right of point E. Formally, it can be said that the portfolio optimization is a process of selecting the best portfolio out of the large set of portfolios considered. The primary aim of portfolio optimization process is the maximization of factors such as expected return E(r) while minimizing the risk associated with it (σ).

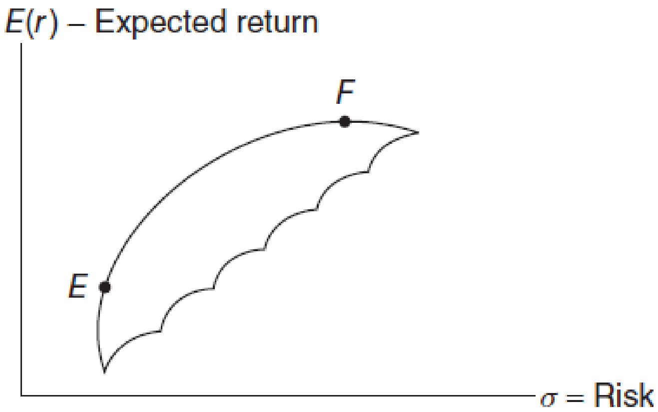

Figure 1 - The Efficient Frontier (Elton E., Brown M., 2015)

2.3 Portfolio Selection

In the third and last category, the “Portfolio Selection” process, the individual creates the utility of returns function which can be formulated into indifference curves as seen in Figure 2. The name indifference curves come from the fact that the curves are constructed in a way which makes the investor equally happy anywhere along the curve. The choices presented along the indifference curve U_3 will be preferred to U_2 and choices along the U_2 will be preferred to U_1. This goes back to the four behavioural assumptions made earlier, where investor prefers more to less. As an example, in Figure 2., the two investors have different utility functions created based on their desired rate of return and acceptable risk which creates two different sets of indifference curves. The first investor based on his set of preferences will choose point A as his preferred portfolio whereas, the second investor will choose portfolio B for his investment. (Elton E., Brown M., 2015)

Changes in the security prices and as the cash dividends are paid out, the risk of each portfolio and its corresponding rate of return is changed on daily basis. This results in constant revaluations of the portfolio analysis which implies that the portfolio management process is a never-ending activity.

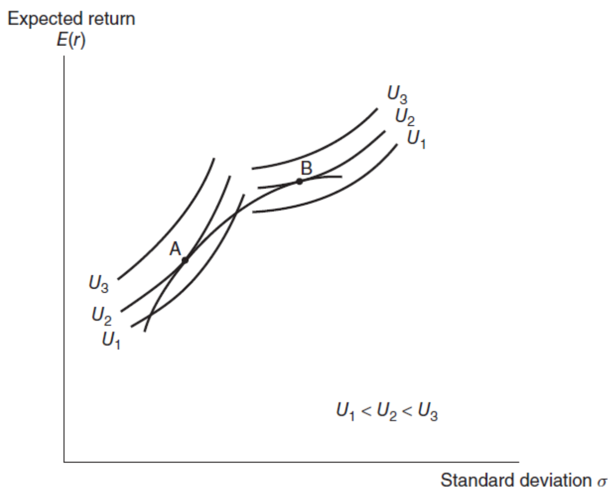

Figure 2 - Set of optimal portfolios (Elton E., Brown M., 2015)

2.4 Benchmark portfolio

As one of the set-out objectives was to create a benchmark portfolio to test the performances of single, multi and industry index models, the portfolio was constructed out of 20 stocks which were chosen from 5 different industry sectors. The stock selection process included two companies in each sector with higher risk/return and two less risky to balance the portfolio. The market index used was S&P500 and the stocks and sectors chosen are as follows:

Sector:	**Company:**
Consumer Staples	Coca-Cola Company (KO) – Soft Drinks
	Walmart (WMT) – Hypermarkets & Super Centres
	Tyson Foods (TSN) – Packaged Foods and Meats
	Kimberly-Clark (KMB) - Household Products
Information Technology	Adobe Inc (ADBE) – Application Software
	Apple (AAPL) – Technology Hardware, Storage and Peripherals
	Microchip Technology (MCHP) – Semiconductors
	Microsoft Corp (MSFT) – System Software
Financial	JPMorgan Chase & Co. (JPM) – Diversified Banks
	Moody's Corp (MCO) – Financial Exchanges and Data

	E-Trade (ETFC) – Investment Banking and Brokerage. (Acquired by Morgan Stanley) Wells Fargo (WFC) – Diversified Banking
	The Travelers Companies Inc. (TRV) – Property and Casualty Insurance
Energy	Chevron Corp (CVX) – Integrated Oil and Gas
	Devon Energy (DVN) – Oil and Gas Exploration & Production
	Marathon Petroleum (MPC) – Oil and Gas Refining and Marketing
	ONEOK (OKE) – Oil and Gas Storage and Transportation
Communication Services	Activision Blizzard (ATVI) – Interactive Home Entertainment
	Facebook (FB) – Interactive Media and Services
	Netflix (NFLX) – Movies and Entertainment
	Omnicom Group (OMC) – Advertising

Table 1 - Companies and sectors used in the benchmark portfolio

Before constructing the portfolio, using python and pandas library, the adjusted closing prices of stocks were plotted to gain insight into their individual performance.

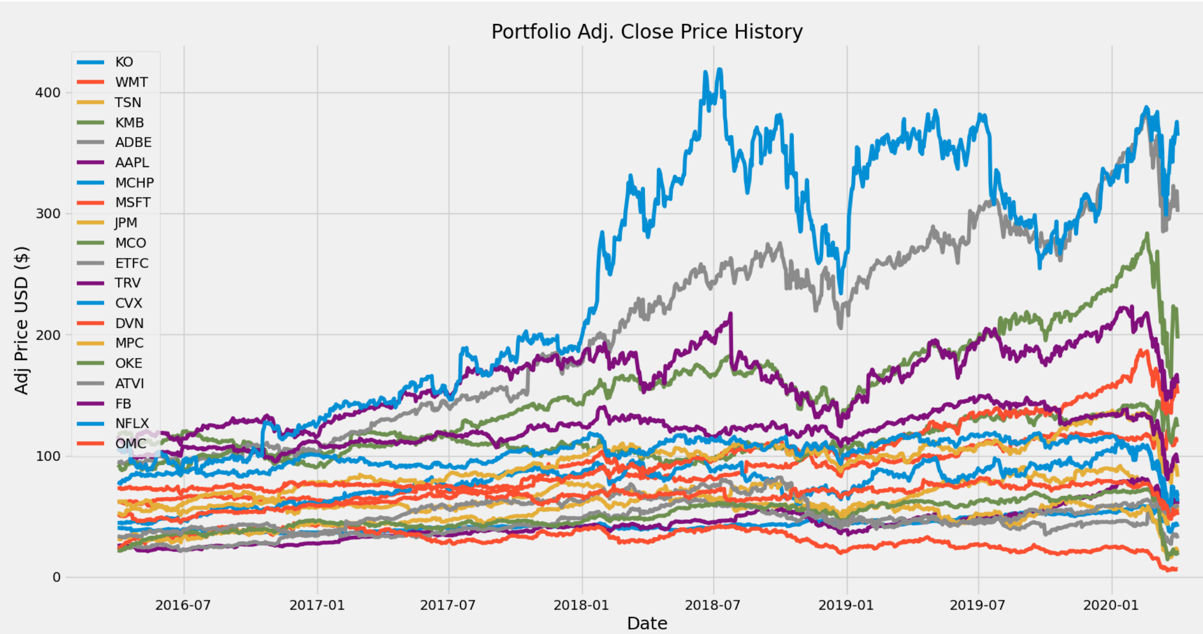

Figure 3 - Adj. Close prices of individual stocks

Using portfolio optimization techniques (Solver in Microsoft Excel), the primary aim was to construct a portfolio with a minimum variance with short-selling allowed. This required the calculation of returns, excess returns and the creation of a var-covar matrix. The resulting weights of stocks for the MVP (Minimum Variance Portfolio) are as follows:

Minimum Variance Portfolio									
KO	WMT	TSN	KMB	ADBE	AAPL	MCHP	MSFT	JPM	MCO
0.0696	0.1099	0.0024	0.2827	0.0967	-0.0149	-0.0549	0.01855	0.01242	-0.1614

Minimum Variance Portfolio									
ETFC	TRV	CVX	DVN	MPC	OKE	ATVI	FB	NFLX	OMC
0.1517	-0.1268	0.3003	-0.1453	-0.0139	-0.0157	0.0843	0.0884	-0.0174	0.1545

Table 2 - Weight of stocks for optimal portfolio

Using the weights from the Table 2. the following formula was applied:

$$\bar{R}_P = \sum_{i=1}^{N} X_i \bar{R}_i$$

Where:

$\bar{R}_P - Expected\ Return\ of\ the\ portfolio$

$X_i - Invested\ fraction\ of\ i'th\ asset$

$\bar{R}_i - Expected\ return\ of\ the\ i'th\ asset$

The following table represents the summary of the annualized performance of the portfolio including the expected return, its variance and risk.

Portfolio Performance	
Annual Expected Return	12.347%
Annual Variance	0.6771%
Annual Standard Deviation (Risk)	8.229%

Table 3 - Portfolio Performance

3. Single Index Model

Casually observing historical stock prices, it can be concluded that in the case of bullish markets (market goes up) there is a tendency for stock prices to follow it, the same phenomenon is present in the case of a bearish market (market goes down) as well. The terms “market goes up” and “market goes down” is referred to some of the many available stock market indexes such as FTSE100 when observing the London Stock Exchange or S&P500 in case of a New York Stock Exchange and many more. Due to this phenomenon, the “Single Index Model” was invented whose underlying assumption is that the rise or fall of stock prices is explained by the changes in the market. (Elton E., Brown M., 2015) Based on the single-index model, the return on a stock “i” can be written as follows:

$$R_i = a_i + \beta_i R_m$$

Equation 4 - Return on the stock (Capinski & Zastawniak, 2011)

Where:

$a_i - the\ component\ of\ security\ i's\ return\ that\ is\ independent\ of\ the\ market\ performance$

$\beta_i - sensitivity\ of\ a\ stock\ return\ to\ the\ change\ in\ the\ market$

$R_m - rate\ of\ return\ on\ the\ market\ index$

The equation of the single index model breaks the return of the stock into two individual components. One component is dependent on market changes and the other one is independent. As stated earlier the Beta variable β_i measures the sensitivity of the stock return to market which means that if β_i has a value of 3 that would indicate that with each 1% change of the market (i.e market goes up by 1%) the stock i would rise by 3%. The second component a_i is generalised as part of the equation that explains the return of the stock independent on the changes of the market. For easier comprehension, the term a_i is broken furtherly into two different components.

$$a_i = \alpha_i + e_i$$

Equation 5 - Component of the security return independent of market performance (Capinski & Zastawniak, 2011)

Where:

$\alpha_i - expected\ value\ of\ a_i$

$e_i - random\ (uncertain)\ element\ of\ a\ with\ the\ expected\ value\ of\ 0$

Taking all of this into account, the equation for the return on the stock i based on single index model becomes:

$$R_i = \alpha_i + \beta_i R_m + e_i$$

Equation 6 - Return on a stock according to single-index model (Capinski & Zastawniak, 2011)

As rate of returns of the market is a random variable as well as e_i which is the uncertain element of a_i, they both represent certain amount of risk which means they both have probability distribution, standard deviation and mean. In this paper, their standard deviations are denoted by σ_m and σ_{ei} and variances as σ_m^2 and σ_{ei}^2.

One assumption of the single-index model that e_i is uncorrelated with R_m which means that the covariance between them should equal to zero or:

$$cov(e_i R_m) = E[(e_i - 0)(R_m - \bar{R}_m)] = 0$$

This can be achieved, if time series-regression analysis is used to obtain the α_i, β_i and σ_{ei}^2. Furthermore, there is one essential assumption made in the single index model construction and that is that the only reason stocks move together, systematically, is because of a common co-movement with the market and that there are no other effects beyond it such as industry or sector effects. Mathematically, this implies that the e_i is independent of e_j for every value of i and j available or written with higher formality $E(e_i e_j) = 0$. Nevertheless, the market changes itself cannot describe in full the stock price movements, for that, there are further factors and more comprehensive models such as industry index model and multi-index model which will be discussed in further detail later on in the report. (Elton E., Brown M., 2015)

Next expression to derive is the expected return on a security. Starting from the basing governing equation of the single index model:

$$E(R_i) = E(\alpha_i + \beta_i R_m + e_i)$$

$$E(R_i) = E(\alpha_i) + E(\beta_i R_m) + E(e_i)$$

To recall on the earlier statements made in this chapter is that the $E(e_i) = 0$ and that α_i , β_i are constants, the equation gets simplified to:

$$E(R_i) = \bar{R}_i = \alpha_i + \beta_i \bar{R}_m$$

Following that, in order to evaluate the risk for the investor, it is important to evaluate that variance of a security and a general expression for it is:

$$\sigma_i^2 = E(R_i - \bar{R}_i)^2$$

Substituting the following expressions:

$$\bar{R}_i = \alpha_i + \beta_i \bar{R}_m \qquad R_i = \alpha_i + \beta_i R_m + e_i$$

$$\sigma_i^2 = E[(\alpha_i + \beta_i R_m + e_i) - (\alpha_i + \beta_i \bar{R}_m)]^2$$

Simplifying the equation by cancelling α_i's and rearing yields to:

$$\sigma_i^2 = E[\beta_i (R_m - \bar{R}_m) + e_i]^2$$

Next step is to square the brackets which results in:

$$\sigma_i^2 = {\beta_i}^2 E(R_m - \bar{R}_m)^2 + 2\beta_i E[e_i (R_m - \bar{R}_m)] + E(e_i)^2$$

Recalling that $E(e_i e_j) = 0$ and $E(e_i)^2 = \sigma_{ie}^2$, the expression for variance of the security under the single index model becomes:

$$\sigma_i^2 = {\beta_i}^2 \sigma_m^2 + \sigma_{ei}^2$$

Lastly, we have the covariance between two securities under the single index model which can be written as:

$$\sigma_{ij} = E[(R_i - \bar{R}_i)(R_j - \bar{R}_j)]$$

Substituting the following expressions:

$$\bar{R}_i = \alpha_i + \beta_i \bar{R}_m \qquad R_i = \alpha_i + \beta_i R_m + e_i$$

$$\bar{R}_j = \alpha_j + \beta_j \bar{R}_m \qquad R_j = \alpha_j + \beta_j R_m + e_j$$

The expression for covariance between two securities yields:

$$\sigma_{ij} = E\{[(\alpha_i + \beta_i R_m + e_i) - (\alpha_i + \beta_i \bar{R}_m)][(\alpha_j + \beta_j R_m + e_j) - (\alpha_j + \beta_j \bar{R}_m)]\}$$

Simplifying the equation by cancelling $\alpha_i{}'s$ and $\alpha_j{}'s$, the expression yields:

$$\sigma_{ij} = E[(\beta_i(R_m - \bar{R}_m) + e_i)(\beta_j(R_m - \bar{R}_m) + e_j)]$$

Multiplying the two brackets results in:

$$\sigma_{ij} = \beta_i \beta_j E(R_m - \bar{R}_m)^2 + \beta_j E[e_i(R_m - \bar{R}_m)] + \beta_i E[e_j(R_m - \bar{R}_m)] + E(e_i e_j)$$

By assumptions made in the single index model, the last three terms are equal to zero, which gives the final expression of the covariance between two securities in the single index model as:

$$\sigma_{ij} = \beta_i \beta_j E(R_m - \bar{R}_m)^2$$

And $E(R_m - \bar{R}_m)^2 = \sigma_m^2$:

$$\sigma_{ij} = \beta_i \beta_j \sigma_m^2$$

3.1 Estimating Beta

Reflecting back on the Equation 6. Where the return of the stock is given with single-index model is given as:

$$R_i = \alpha_i + \beta_i R_m + e_i$$

This equation is expected to hold true at all points in time, regardless of the fact that values of α_i, β_i and σ_{ei}^2 will change constantly. These values cannot simply be observed but are estimated from the historical data of the individual stock returns and the market returns. Also, it must be highlighted that Equation 6. is a straight-line equation, so, if the σ_{ei}^2 yielded to zero, then the values of α_i and β_i would simply be estimated from two observations. Nevertheless, the presence of a random element e_i means that the results will scatter as points around the straight line as can be seen in Figure 4. The greater the value of the random element, the points will scatter further away from the straight line and as we actually don't know where the straight line is after getting the points, the method of estimating it is done using linear regression analysis. (Elton E., Brown M., 2017) Lastly, to point out that the scatter of points is represented on the x-y coordinate system where the x-axis represents the return of the market and the y-axis return on the stock i, whereas, the time period of plots must be equal in both measures (If monthly returns on the stock i are used, then consequently, monthly returns on the market must be used).

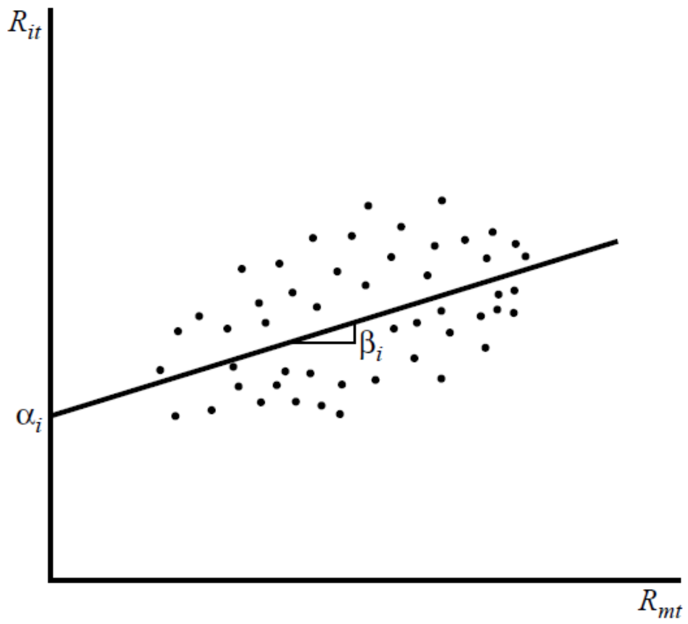

Figure 4 - Returns of the security vs market return (Elton E., Brown M., 2017)

From a scatter plot like this one in Figure 3. the formal equation for obtaining Beta using regression analysis in a time period from t=1 to t=N is as follows:

$$\beta_i = \frac{\sigma_{im}}{\sigma_m^2} = \frac{\sum_{t=1}^{N}[(R_{it} - \bar{R}_{it})(R_{mt} - \bar{R}_{mt})]}{\sum_{t=1}^{N}(R_{mt} - \bar{R}_{mt})^2}$$

Equation 7 - Beta using linear regression analysis.

After obtaining beta, the estimation of alpha comes from:

$$\alpha_i = \bar{R}_i - \beta_i \bar{R}_m$$

Equation 8 - Alpha using linear regression analysis.

3.2 Portfolio construction using Single Index model

If an assumption is made that all of the securities in an investor’s portfolio follow the single-index model in terms that their respected return, variance and covariances between them can be calculated using the results from the single-index model, then, the single-index model can be applied to a portfolio. (Francis, 2013) For the successful application, the following input parameters are required for an N asset portfolio:

- N values of α
- N values of β
- N values of σ_{ei}^2
- Expected return $\bar{R}_m$ of the market and its variance σ_m^2

For the construction of the portfolio using the single index model, it must be noted that the advantage of doing so is that the single index model simplifies the risk and return calculations compared to using “var-covar” matrix. To achieve this, the equation for the expected return for a portfolio using a single index model must be stated but firstly, the standard equation used for calculation of the expected return of a portfolio is:

$$\bar{R}_P = \sum_{i=1}^{N} X_i \bar{R}_i$$

Equation 9 - Portfolio Expected Return (Francis, 2013)

Applying the single index model to it results in:

$$\bar{R}_P = \sum_{i=1}^{N} X_i(\alpha_i + \beta_i \bar{R}_m) = \sum_{i=1}^{N} X_i \alpha_i + \sum_{i=1}^{N} X_i \beta_i \bar{R}_m$$

When applying the single index model to a portfolio, both α_P and β_P are defined as the weighted average of the individual $\alpha's$ and $\beta's$ of the securities. The equations are as follows:

$$\alpha_P = \sum_{i=1}^{N} X_i \alpha_i$$

Equation 10 - Alpha of a portfolio (Francis, 2013)

$$\beta_P = \sum_{i=1}^{N} X_i \beta_i$$

Equation 11 - Beta of a portfolio (Francis, 2013)

Combining the definitions of portfolio alphas and betas, the final expected return of a portfolio results in:

$$\bar{R}_P = \alpha_P + \beta_P \bar{R}_m$$

Equation 12 - Portfolio Expected Return based on a single index model

Next parameter to look at is the portfolio variance based on the single-index model but first, the general equation of a portfolio variance is:

$$\sigma_P^2 = \sum_{i=1}^{N} X_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{j=1}^{N} X_i X_j \sigma_{ij}$$

Equation 13 - Standard Equation for Portfolio Variance

Applying that standard equation to the single-index model results in:

$$\sigma_P^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} X_i X_j \beta_i \beta_j \sigma_m^2 + \sum_{i=1}^{N} X_i^2 \sigma_{ie}^2$$

Next step is to simplify:

$$\sigma_P^2 = \left(\sum_{i=1}^{N} X_i \, \beta_i\right)\left(\sum_{j=1}^{N} X_j \, \beta_j\right)\sigma_m^2 + \sum_{i=1}^{N} X_i^2 \, \sigma_{ie}^2$$

Applying the fact that portfolio beta is:

$$\beta_P = \sum_{i=1}^{N} X_i \, \beta_i$$

Equation 14 - Portfolio Beta (Francis, 2013)

And assuming that portfolio has a lot of securities so the $\sum_{i=1}^{N} X_i^2 \, \sigma_{ie}^2$ term becomes negligible as the risk gets diversified away, the final equation for the variance of the portfolio based on the single index model becomes:

$$\sigma_P^2 = \beta_P^2 \sigma_m^2$$

Equation 15 - Variance of the portfolio based on the single-index model (with assumption)

Or without assuming:

$$\sigma_P^2 = \beta_P^2 \sigma_m^2 + \sum_{i=1}^{N} X_i^2 \, \sigma_{ie}^2$$

Equation 16 - Variance of the portfolio based on the single-index model (without assumption)

Standard deviation then becomes:

$$\sigma_P = \beta_P \sigma_m$$

Equation 17 - Standard deviation of the portfolio based on the single-index model

The term σ_{ie}^2 disappears from the standard deviation equation as the portfolio gets larger. That is because the σ_{ie}^2 term is considered as “diversifiable risk” which means it can be eliminated by investing in lots of different securities while the beta is a measure of a non-diversifiable risk. (Francis, 2013)

3.3 Running the model

To perform the analysis on the benchmark portfolio, we also had to utilize a certain timeframe. The current model will be analysed between 30/04/2016 to 31/03/2020. Using python as a programming language, the analysis of the benchmark portfolio was conducted firstly by analysing the returns (the returns of each individual stock are located in Appendix B) and then evaluating alphas, betas, variances and covariances for each individual stock, whose results can be found in the table below:

Table 4 – Results of individual securities

Company:	Alpha:	Beta:	Variance:	Covariance with market:	Expected Return:
KO	-0.000549	0.594765	0.001909	0.000906	0.002728
WMT	0.011681	0.408405	0.002487	0.000622	0.013931
TSN	-0.002459	0.834231	0.007049	0.001271	0.002138
KMB	0.000507	0.411868	0.002373	0.000627	0.002777
ABDE	0.022237	0.954476	0.003603	0.001454	0.027498
AAPL	0.016223	1.109522	0.006394	0.001690	0.022337
MCHP	0.004011	1.647879	0.008714	0.002511	0.013093
MSFT	0.020283	0.833168	0.002361	0.001269	0.024874
JPM	0.005907	1.399430	0.005023	0.002132	0.013619
MCO	0.012677	1.258877	0.004240	0.001918	0.019614
ETFC	0.004839	1.161935	0.007474	0.001770	0.011242
TRV	-0.005036	0.919294	0.002882	0.001400	0.000029
CVX	-0.006449	1.089588	0.003801	0.001660	-0.000445
DVN	-0.029135	2.827000	0.023629	0.004308	-0.013556
MPC	-0.010289	2.111983	0.013741	0.003218	0.001349
OKE	-0.001446	2.082159	0.015995	0.003173	0.010028
ATVI	0.012707	0.600966	0.007358	0.000915	0.016019
FB	0.004231	1.174103	0.005860	0.001789	0.010701
NFLX	0.028197	0.902641	0.012489	0.001375	0.033172
OMC	-0.008479	0.731549	0.003167	0.001114	-0.004448
Market (S&P500)	-	-	0.001524	-	0.005510

With the obtained results for the weight of stocks of the optimal portfolio, it is possible to utilize equations 10,11,12 and 16 to evaluate portfolio alpha, beta, expected return and risk.

$$\beta_P = \sum_{i=1}^{N} X_i \beta_i = 0.9174653$$

$$\alpha_P = \sum_{i=1}^{N} X_i \alpha_i = 0.00569321$$

Now, after acquiring the portfolio alphas and betas we can evaluate the portfolio expected monthly return and its risk.

$$\bar{R}_P = \sum_{i=1}^{N} X_i(\alpha_i + \beta_i \bar{R}_m) = \sum_{i=1}^{N} X_i \alpha_i + \sum_{i=1}^{N} X_i \beta_i \bar{R}_m$$

$$\bar{R}_P = 0.00569321 + 0.9174653 \times 0.00551088163$$

$$\bar{R}_P = 0.0107493$$

$$\sigma_P^2 = \beta_P^2 \sigma_m^2 + \sum_{i=1}^{N} X_i^2 \sigma_{ie}^2$$

$$\sigma_P^2 = 0.9174653^2 \times 0.001524 + (0.0908^2 \times 0.00137) + \dots + (0.014^2 \times 0.0023515)$$

$$\sigma_P^2 = 0.0014741076$$

$$\sigma_P = 0.040122$$

3.4 Performance of the Single Index Model

As stated out earlier, the single index model has the underlying assumption that the rise or fall of stock prices is explained by the changes in the market. To understand how much of the stock price rises and falls were explained by the market movement and essentially, to evaluate the performance of the single index model on stock-by-stock basis, the correlation coefficient was calculated for each security in the portfolio. The formula for correlation coefficient is as follows:

$$\rho = \frac{\sigma_{ij}}{\sigma_i \sigma_j}$$

Where in our case:

$\rho - Correlation\ coefficient$

$\sigma_{ij} - Covariance\ between\ actual\ returns\ and\ returns\ predicted\ by\ the\ model$

$\sigma_i - Standard\ deviation\ of\ the\ actual\ returns$

$\sigma_j - Standard\ deviation\ of\ the\ returns\ predicted\ by\ the\ model$

For the single index model, the results are as follows:

Table 5 - Performance of the single index model

Correlation Results						
	KO	**WMT**	**TSN**	**KMB**	**ADBE**	**AAPL**
σ_{ij}	0.00054	0.00025	0.00106	0.00026	0.00139	0.00188
σ_i	0.04369	0.04987	0.08396	0.04872	0.06003	0.07996
σ_j	0.02322	0.01594	0.03257	0.01608	0.03726	0.04331
ρ	0.531449	0.319719	0.330029	0.620743	0.541656	0.541656

	MCHP	**MSFT**	**JPM**	**MCO**	**ETFC**	**TRV**	**CVX**
σ_{ij}	0.00414	0.00106	0.00298	0.00242	0.00206	0.00129	0.00181
σ_i	0.09335	0.04859	0.07087	0.06511	0.08645	0.05368	0.06166
σ_j	0.06433	0.03253	0.05463	0.04914	0.04536	0.03589	0.04254
ρ	0.689	0.669	0.771	0.755	0.525	0.669	0.69

	DVN	**MPC**	**OKE**	**ATVI**	**FB**	**NFLX**	**OMC**
σ_{ij}	0.0122	0.0068	0.0066	0.0006	0.0021	0.0012	0.0008
σ_i	0.1537	0.1172	0.1265	0.0858	0.0766	0.1118	0.0563
σ_j	0.1104	0.0824	0.0813	0.0235	0.0458	0.0352	0.0286
ρ	0.717943	0.703355	0.642709	0.273503	0.598726	0.315312	0.507463

Single-index model for some stocks explained as much as 77% of its rises and falls in the 4-year period. This is very promising as it provides the investor with a great probability of predicting future stock returns. Nevertheless, the correlation coefficient between the actual returns and predicted returns on some stocks was as low as 27%, which is not as efficient as expected. The performance of the single index model is visually represented in the Figure 5.

Figure 5 - Performance of the single index model

4. Multi-index model

In the previous section we have examined a single-index model which assumes that the only reason stocks move together is because of their common co-movement with the market. Obviously, there are more reasons why security prices move together and that's what multi-index model is trying to capture. To do this, an individual must find a set of economic factors that can account for common movement in stock prices beyond what is explained by the market movement. This is not very difficult to do, but what one must realize, is that the new set of economic factors that explains co-movement between the stock prices must not be correlated to the market movement. The influence of other non-market factors was first introduced in 1966 from which two types of multi-index models were developed: General Multi-index model and Industry-index model. (Elton E., Brown M., 2017)

4.1 General Multi-index Model

Building upon the equation of the single-index model, additional indices can be introduced by simply adding these influences on the general return equation. For example:

$$R_i = a_i^* + b_{i1}^* I_1^* + b_{i2}^* I_2^* + \cdots + b_{iL}^* I_L^* + c_i$$

Equation 18 - Return on the stock based on multi-index model (Elton E., Brown M., 2017)

Where:

$a_i^* - expected\ value\ of\ unique\ return$

$I_1^* - actual\ level\ of\ the\ first\ index$

$b_i^* - responsivness\ of\ the\ return\ on\ stock\ i\ to\ changes\ in\ the\ index\ value$

$c_i - random\ component\ of\ the\ unique\ return$

Although this approach to implementing multi-index model by simply adding another factor and the responsiveness of the security to that factor doesn't seem too difficult but we must recall the earlier highlighted point about multi-index models. To achieve desirable mathematical properties for the multi-index model, the indices used should be uncorrelated (orthogonal). Luckily, this brings no theoretical problems as any two indexes can be made uncorrelated. (Elton E., Brown M., 2017) As an example, a two-index model is presented to illustrate the procedure for uncorrelation. Note that this can be done for any number of indexes used in the model (3, 4, 5, …, N).

Starting with the general equation for the return of a two-index model:

$$R_i = a_i^* + b_{i1}^* I_1^* + b_{i2}^* I_2^* + c_i$$

Assuming that the first index I_2^* represents the market and I_2^* could be a sector index. The first step required to make these two uncorrelated is to perform the regression analysis. But first, let's assume that $I_1 = I_1^*$.

$$I_2^* = \gamma_0 + \gamma_1 I_1 + d_t$$

Where:

$\gamma_0, \gamma_1 - Regression\ coefficients$

$d_t - Random\ Error\ term$

Now in the regression analysis, this makes d_t uncorrelated to I_1, so this means that:

$$d_t = I_2^* - (\gamma_0 + \gamma_1 I_1)$$

Is actually an index of the sector index performance independent of the market (uncorrelated). So:

$$I_2 = d_t = I_2^* - \gamma_0 - \gamma_1 I_1$$

Where I_2 represents the index of sector performance (uncorrelated to the market) what was the original purpose of regression analysis. Solving the above equation for I_2^* to substitute in the general return equation of a two-index model, the result is:

$$I_2^* = I_2 + \gamma_0 + \gamma_1 I_1$$

$$R_i = a_i^* + b_{i1}^* I_1 + b_{i2}^*(I_2 + \gamma_0 + \gamma_1 I_1) + c_i$$

$$R_i = a_i^* + b_{i1}^* I_1 + b_{i2}^* I_2 + b_{i2}^* \gamma_0 + b_{i2}^* \gamma_1 I_1 + c_i$$

Rearranging the terms yields:

$$R_i = (a_i^* + b_{i2}^* \gamma_0) + (b_{i1}^* + b_{i2}^* \gamma_1) I_1 + b_{i2}^* I_2 + c_i$$

As the first bracket in the equation is a constant it can be simply defined as a_i. As b_{i1}^* and b_{i1}^* are constants, they also, can be simply defined as b_{i1} and b_{i2} respectfully. The equation is the simplified to:

$$R_i = a_i + b_{i1} I_1 + b_{i2} I_2 + c_i$$

Equation 19 - Return on the security i based on the two-index model with uncorrelated indexes. (Elton E., Brown M., 2017)

Now that we have a general equation for the return of the security "i" with two-index model where the indexes are uncorrelated, we can simply apply the same technique for a N number of indexes in the model and obtain the general equation for the return:

$$R_i = a_i + b_{i1} I_1 + b_{i2} I_2 + b_{i3} I_3 + \cdots + b_{iL} I_L + c_i$$

Equation 20 - Return on the security i based on the multi-index model with uncorrelated indexes (Elton E., Brown M., 2017)

As well as the single-index model, the multi-index model holds some underlying assumptions and the assumption this model is making is that the only reason stocks move together is because of their common co-movement with the set indexes used in the model. Or mathematically written as $E(c_i c_j) = 0$ and this holds for all stocks where i=1, 2, …, N and j = 1, 2, …, N but i≠j.

Furthermore, the expected return, variance and covariance among different securities based on multi-index model are as follows:

Expected Return:

$$\bar{R}_i = a_i + b_{i1}\overline{I_1} + b_{i2}\overline{I_2} + \cdots + b_{iL}\overline{I_L}$$

Equation 21 - Expected Return (Elton E., Brown M., 2017)

Variance of Return:

$$\sigma_i^2 = b_{i1}^2\sigma_{I1}^2 + b_{i2}^2\sigma_{I2}^2 + \cdots + b_{iL}^2\sigma_{IL}^2 + \sigma_{ci}^2$$

Equation 22 - Variance of return (Elton E., Brown M., 2017)

Covariance between security i and j:

$$\sigma_{ij} = b_{i1}b_{j1}\sigma_{I1}^2 + b_{i2}b_{j2}\sigma_{I2}^2 + \cdots + b_{iL}b_{jL}\sigma_{IL}^2$$

Equation 23 - Covariance between two securities (Elton E., Brown M., 2017)

4.2 Industry Index Model

Industry-index model is a simple two-index model which builds on the single-index model by adding the industry index to capture the effect it has on the co-movement between the securities. Note that the industry index must be uncorrelated to the market via regression analysis. (Elton E., Brown M., 2017) This model is represented as follows:

$$R_i = a_i + b_{im}I_m + b_{i1}I_1 + c_i$$

Equation 24 - Industry index model

Note that industry index model can include more than one industry but then care must be taken to make sure that indexes are not only uncorrelated with the market movement but uncorrelated with each other as well. This yields to a general equation for the industry index model:

$$R_i = a_i + b_{im}I_m + b_{i1}I_1 + b_{i2}I_2 + \cdots + b_{iL}I_L + c_i$$

Where:

$I_m - Market\ index$

$I_i - Industry\ indexes\ that\ are\ uncorrelated\ with\ the\ market\ and\ each\ other$

4.3 Implementing the Multi Index model

The same formula for the industry index model can be applied to a sector index model whereas, instead of the industry beta and index, the sector beta and index are chosen. Note that this is not implementation of the sector index model, because the sector index is applied to all stocks. The sector selected was the S&P Financial Select Sector (XLF) and it was implemented under the assumption that the indices of the market and the sector are already uncorrelated. The process of linear regression in purpose of uncorrelation will be done in a multi-index model where there are more indices added to the equation. The results on individual security basis can be found in the Table 4.

Company:	Alpha:	Beta Market:	Beta Sector:	Variance:	Expected Return:
KO	-0.0005	0.594765	0.204642	0.001909	0.002728
WMT	0.0116	0.408405	0.116111	0.002487	0.013931
TSN	-0.0024	0.834231	0.264753	0.007049	0.002138
KMB	0.00050	0.411868	0.081132	0.002373	0.002777
ABDE	0.02223	0.954476	0.337968	0.003603	0.027498
AAPL	0.01622	1.109522	0.382183	0.006394	0.022337
MCHP	0.00401	1.647879	0.739426	0.008714	0.013093
MSFT	0.02028	0.833168	0.253227	0.002361	0.024874
JPM	0.00590	1.399430	0.797104	0.005023	0.013619
MCO	0.01267	1.258877	0.436148	0.004240	0.019614
ETFC	0.00483	1.161935	0.825926	0.007474	0.011242
TRV	-0.00503	0.919294	0.389442	0.002882	0.000029
CVX	-0.00644	1.089588	0.557649	0.003801	-0.000445
DVN	-0.02913	2.827000	1.413069	0.023629	-0.013556
MPC	-0.01028	2.111983	0.995235	0.013741	0.001349
OKE	-0.00144	2.082159	1.136705	0.015995	0.010028
ATVI	0.012707	0.600966	0.092526	0.007358	0.016019
FB	0.004231	1.174103	0.421680	0.005860	0.010701
NFLX	0.028197	0.902641	0.212675	0.012489	0.033172
OMC	-0.008479	0.731549	0.428920	0.003167	-0.004448
Market (S&P500)	-	-	-	0.001524	0.005510
S&P Financial Select Sector (XLF)	-	-	-	0.004737	0.012179

Table 6 - Results of individual securities for Industry-index model

Constructing a portfolio using sector-index model requires determining of two different portfolio betas, one in reference to the market and another in in reference to the sector. Fortunately, the beta of the portfolio in reference to the market has already been determined in the implementation of the single-index model, assuming that we keep the weights of each stock in the portfolio the same. So, the information that is already available is as follows:

$$\alpha_P = \sum_{i=1}^{N} X_i \alpha_i = 0.00569321$$

And

$$\beta_{P,M} = \sum_{i=1}^{N} X_i\,\beta_i = 0.9174653$$

$$\beta_{P,S} = \sum_{i=1}^{N} X_i\,\beta_j = 0.385273$$

Expected monthly return of the sector index model is then:

$$\bar{R}_P = \sum_{i=1}^{N} X_i\alpha_i + \sum_{i=1}^{N} X_i\,\beta_i\bar{R}_m + \sum_{i=1}^{N} X_i\,\beta_j\bar{R}_s$$

$$\bar{R}_P = 0.00569321 + 0.9174653 \times 0.00551088163 + 0.385273 \times 0.01217996532$$

$$\bar{R}_P = 0.01544186$$

And the risk associated is:

$$\sigma_P^2 = \beta_P^2\sigma_m^2 + \beta_S^2\sigma_s^2 + \sum_{i=1}^{N} X_i^2\,\sigma_{ie}^2$$

Knowing that:

$$\beta_P^2\sigma_m^2 + \sum_{i=1}^{N} X_i^2\,\sigma_{ie}^2 = 0.0014741076$$

Risk of the portfolio becomes:

$$\sigma_P^2 = 0.0014741076 + 0.385273^2 \times 0.04737$$

$$\sigma_P^2 = 0.0085055$$

4.4 Performance of the Multi Index Model

To test the performance of the multi-index model where a sector index was added in the equation, it was logical to test multiple different sectors to find how much they value they add the model performance. The sectors used were: S&P Consumer Staples Select Sector (XLP), S&P Financial Select Sector (XLF) and S&P Technology Select Sector (XLK).

4.4.1 Multi Index Model with Consumer Index

As for the single index model, the results of covariance between actual results and results predicted by the multi-index model ($\boldsymbol{\sigma_{ij}}$), standard deviation of the actual returns ($\boldsymbol{\sigma_i}$), and standard deviation of the returns predicted by the multi index model ($\boldsymbol{\sigma_j}$), are presented in the Table 7.

Table 7 - Performance of the multi index model with consumer index

Correlation Results						
	KO	**WMT**	**TSN**	**KMB**	**ADBE**	**AAPL**
σ_{ij}	0.00146	0.00098	0.00257	0.00105	0.00158	0.00263
σ_i	0.04369	0.04987	0.08396	0.04872	0.06003	0.07996
σ_j	0.04892	0.03939	0.06527	0.0406	0.04805	0.06506
ρ	0.689	0.505	0.475	0.536	0.553	0.51

	MCHP	**MSFT**	**JPM**	**MCO**	**ETFC**	**TRV**	**CVX**
σ_{ij}	0.00492	0.00136	0.00367	0.00303	0.00203	0.0022	0.00257
σ_i	0.09335	0.04859	0.07087	0.06511	0.08645	0.05368	0.06166
σ_j	0.08617	0.04623	0.07511	0.06857	0.04723	0.06039	0.06444
ρ	0.618	0.613	0.697	0.685	0.501	0.685	0.652

	DVN	**MPC**	**OKE**	**ATVI**	**FB**	**NFLX**	**OMC**
σ_{ij}	0.01491	0.00887	0.00844	0.0009	0.00259	0.00123	0.00127
σ_i	0.1537	0.1172	0.1265	0.0858	0.0766	0.1118	0.0563
σ_j	0.15126	0.11836	0.11496	0.03862	0.06312	0.03759	0.04566
ρ	0.648	0.646	0.587	0.275	0.541	0.296	0.498

4.4.2 Multi Index Model with Financial Index

The results of the multi index model with financial index are presented in the Table 8.

Table 8 - Performance of the multi index model with financial index

Correlation Results						
	KO	**WMT**	**TSN**	**KMB**	**ADBE**	**AAPL**
σ_{ij}	0.00059	0.00027	0.00142	0.00024	0.00118	0.00204
σ_i	0.04369	0.04987	0.08396	0.04872	0.06003	0.07996
σ_j	0.02783	0.01821	0.03819	0.01725	0.04504	0.05194
ρ	0.491	0.301	0.449	0.288	0.44	0.498

	MCHP	**MSFT**	**JPM**	**MCO**	**ETFC**	**TRV**	**CVX**
σ_{ij}	0.00427	0.00097	0.0035	0.0021	0.0016	0.00139	0.00229
σ_i	0.09335	0.04859	0.07087	0.06511	0.08645	0.05368	0.06166
σ_j	0.08442	0.03773	0.07985	0.05903	0.07501	0.04607	0.05905
ρ	0.547	0.533	0.625	0.553	0.249	0.568	0.636

	DVN	MPC	OKE	ATVI	FB	NFLX	OMC
σ_{ij}	0.016	0.00868	0.00812	0.00057	0.00225	0.00119	0.00109
σ_i	0.1537	0.1172	0.1265	0.0858	0.0766	0.1118	0.0563
σ_j	0.15157	0.11038	0.11632	0.02453	0.05565	0.03881	0.04237
ρ	0.694	0.678	0.558	0.275	0.535	0.276	0.462

4.4.3 Multi Index Model with Technology Index

The results of the multi index model with technology index are presented in the Table 9.

Table 9 - Performance of the multi index model with technology index

Correlation Results						
	KO	WMT	TSN	KMB	ADBE	AAPL
σ_{ij}	0.00077	0.00048	0.00178	0.00038	0.00349	0.00594
σ_i	0.04369	0.04987	0.08396	0.04872	0.06003	0.07996
σ_j	0.03726	0.02975	0.0573	0.02609	0.08016	0.10327
ρ	0.4787	0.3241	0.373	0.2993	0.7334	0.7266

	MCHP	MSFT	JPM	MCO	ETFC	TRV	CVX
σ_{ij}	0.00777	0.00272	0.00431	0.00484	0.00313	0.00196	0.00228
σ_i	0.09335	0.04859	0.07087	0.06511	0.08645	0.05368	0.06166
σ_j	0.1202	0.07068	0.08814	0.09494	0.07552	0.05987	0.06262
ρ	0.699	0.8	0.697	0.792	0.484	0.617	0.597

	DVN	MPC	OKE	ATVI	FB	NFLX	OMC
σ_{ij}	0.01762	0.01008	0.00914	0.00205	0.00455	0.00407	0.00103
σ_i	0.1537	0.1172	0.1265	0.0858	0.0766	0.1118	0.0563
σ_j	0.1784	0.13531	0.12771	0.06	0.09195	0.08535	0.04192
ρ	0.649	0.642	0.572	0.402	0.654	0.432	0.439

4.4.4 Comparison

In the Figure 6. a visual representation of the model performance (including the single index model) is presented. It can be seen that multi index model with technology sector index explains the most price movements for the companies in that sector. So, to get a better understanding of the models performance, next step is to show the model performance sector by sector.

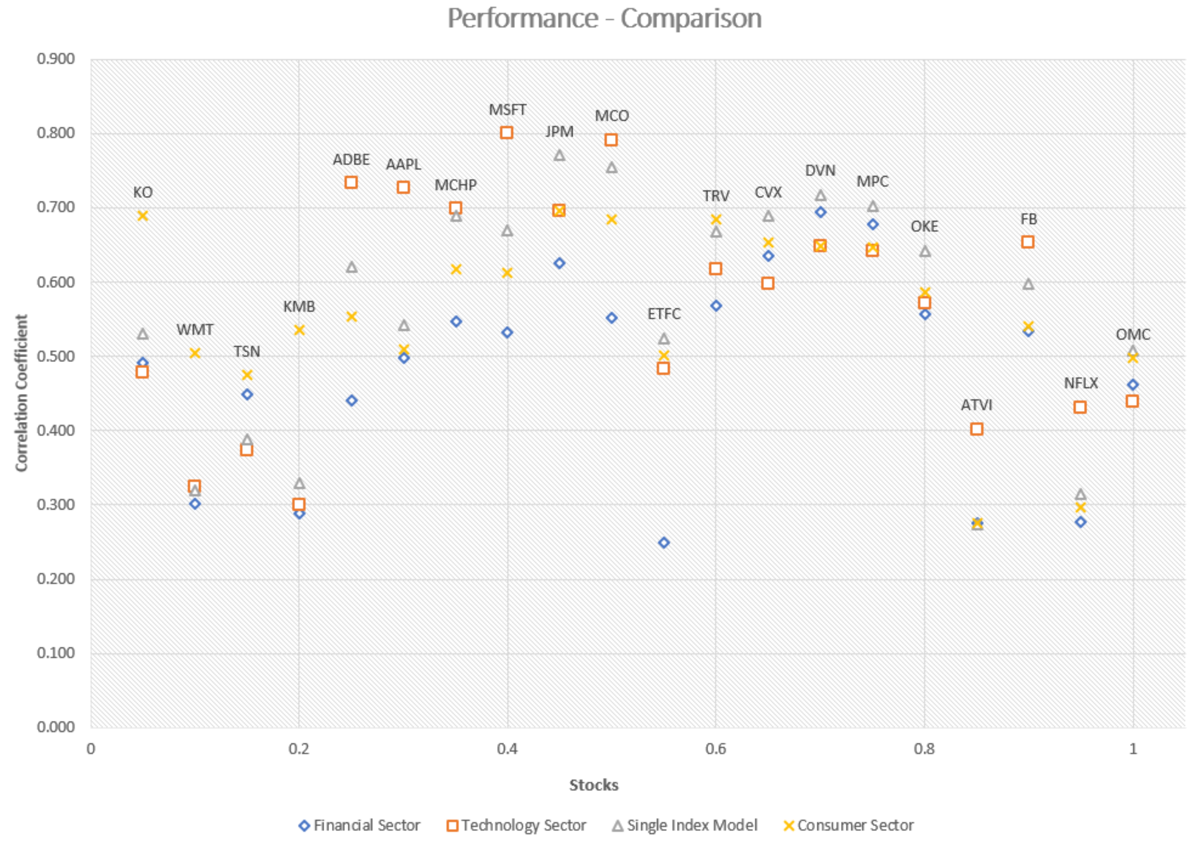

Figure 6 - Comparison of the performance of each model

For the following four stocks in the portfolio: Coca-Cola (KO), Walmart (WMT), Tyson Foods (TSN) and Kimberly-Clark (KMB) which are all in the consumer staples sector, the performance between the single-index model and multi index model with consumer sector index included is represented in the Figure 7. It can be clearly concluded that by including the consumer sector index increases the overall performance of the model for the stocks in that sector.

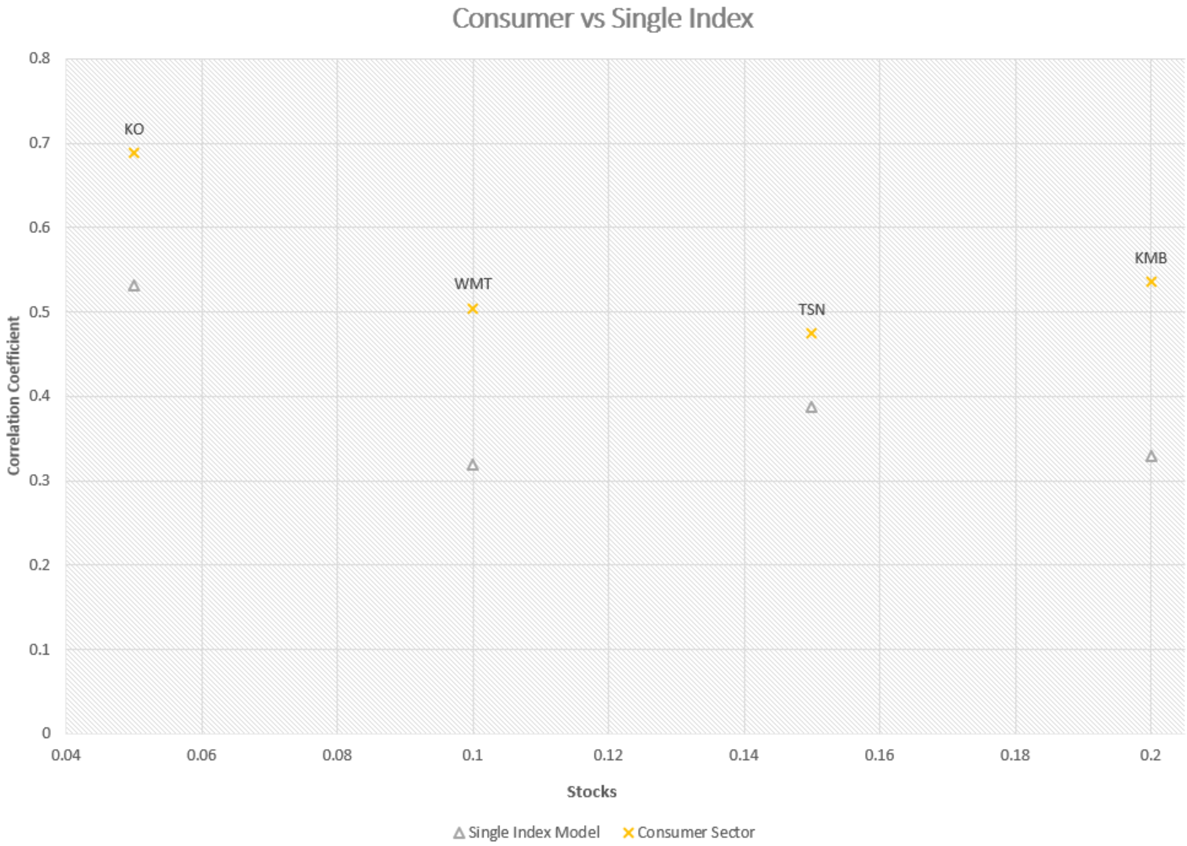

Figure 7 - Performance comparison for stocks in consumer sector

The next four stocks in the portfolio: JPMorgan Chase & Co (JPM), Moody's Corp (MCO), E-Trade (ETFC) and The Travelers Companies Inc. (TRV) which are all part of the financial sector, the performance between the single index model and multi-index model with financial sector added to the equation is represented in Figure 8. Surprisingly, the single index model has outperformed the multi-index model with financial sector index for the E-Trade company, which is unusual as single index model underperformed significantly in all other financial sector companies.

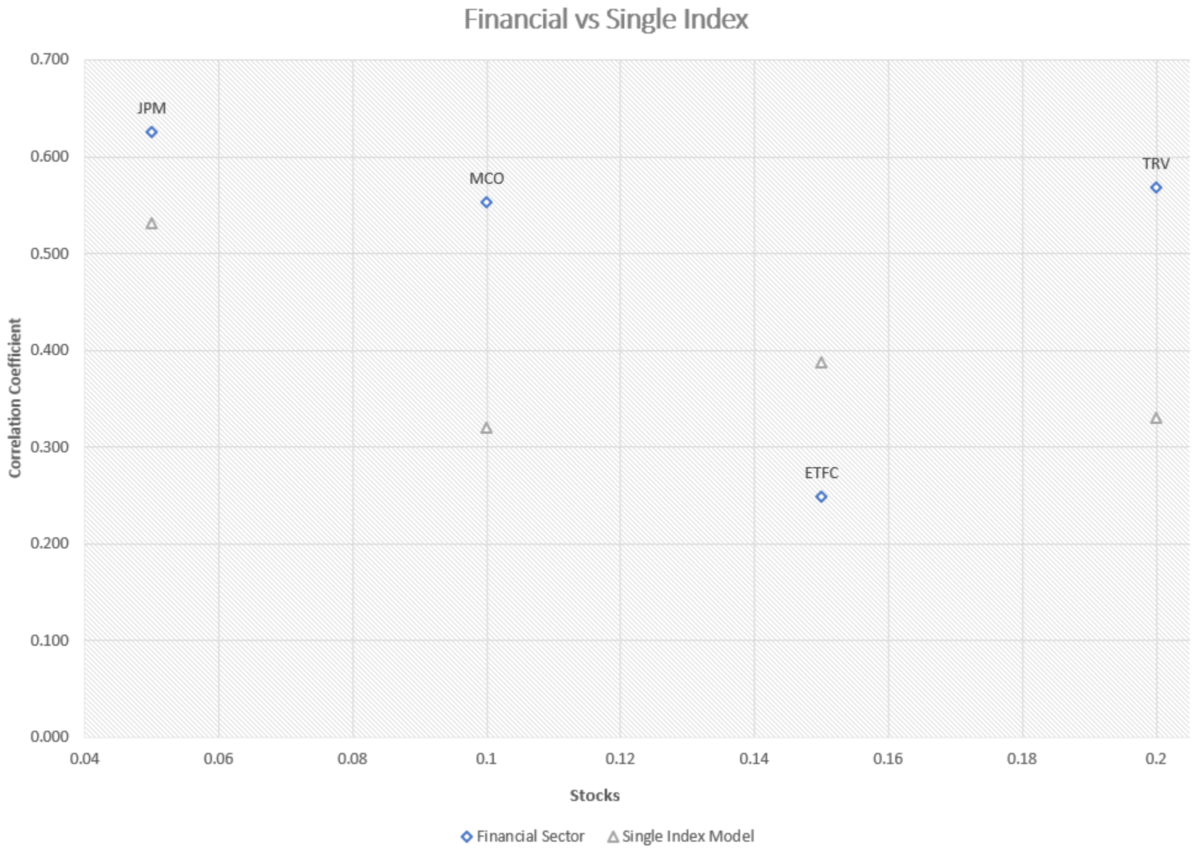

Figure 8 - Performance comparison for stocks in financial sector

Lastly, we look at the technology sector with the stocks: Adobe Inc (ADBE), Apple (AAPL), Microchip Technology (MCHP) and Microsoft Corp (MSFT). The performance is presented in Figure 9. and we see that the multi-index model with technology sector index is by far the highest performing. The correlation coefficients reach values of up to 0.8 for Microsoft which provides significant probability of the model predicting future price movements.

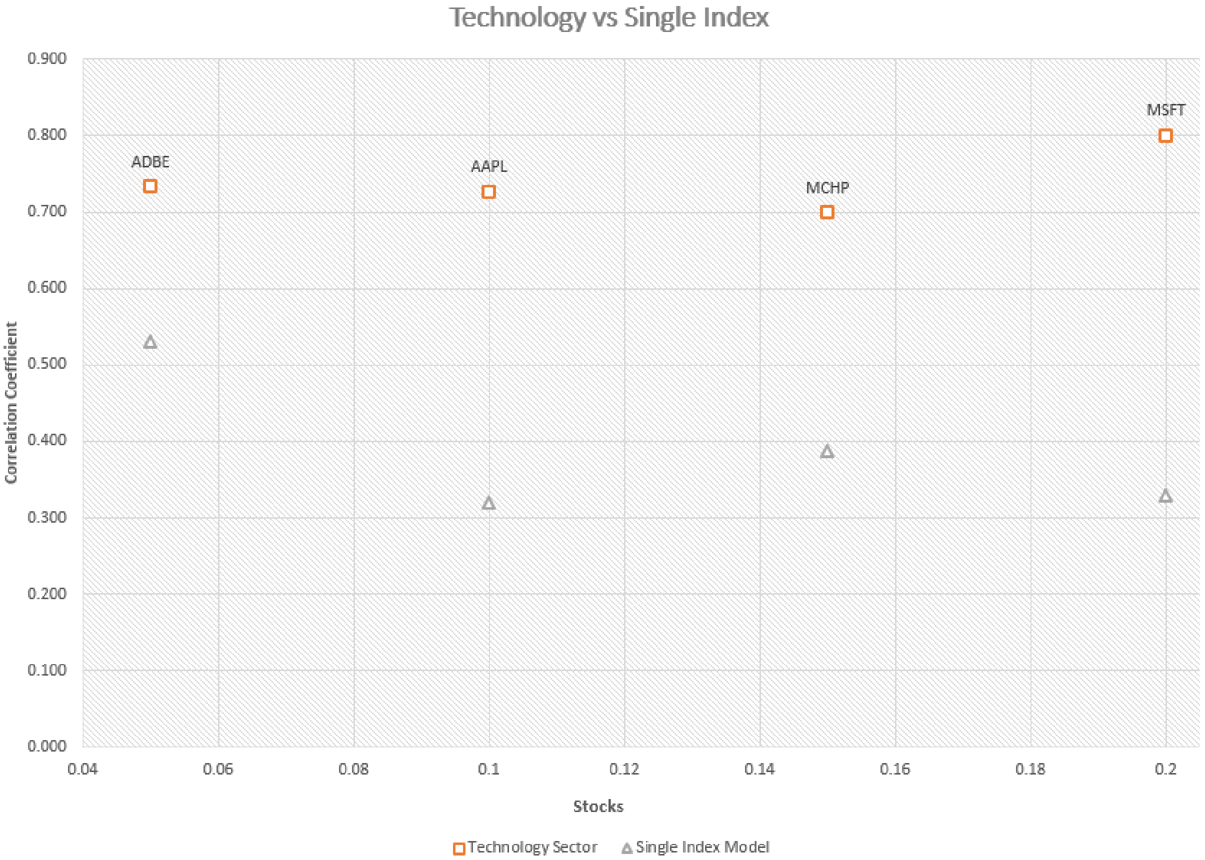

Figure 9 - Performance comparison for stocks in technology sector

5. Fundamental Multi-index Models

When discussing the expansion of the single-index model and the addition of extra indices to explain share price movements (i.e., creation of a multi-index models), there are two fundamental models that have gotten a great deal of academic and practical attention. These two models were first introduced by Fama and French (1993) and Chen, Roll, and Ross (1986).

5.1 Fama-French Three Factor Model

Building on the single-index model firstly introduced in early 1960s by William Sharpe and John Lintner (W. Sharpe, 1963), Fama and French introduced the groundwork for a multi-index model which was based on the premise that company size (market capitalization) and the ratio of book value of equity to the market value of equity have a strong impact on the average return of common stocks. Now, the first problem with this is the fact that the components of their model, such as the book value of equity, is reported at most four times a year and in order for testing of time series to exist, monthly observations are required at least. In order to overcome this, Fama and French formed portfolios whose returns mimic the impact of the said variables. Doing this, allowed Fama and French to convert set of variables that couldn't be observed at frequent intervals into a set of traded assets that have prices and returns that can be observed at any moment of time and over any interval. The construction of these variables is a two-step process.

Step 1: Define two groups of stocks based on the company size (which is determined as price of the stock times the number of shares) where the first group contains all stocks on the NYSE, NASDAQ, and AMEX with a company size larger than the median size of a stock on the NYSE. The second group of stocks is formed of all smaller stocks that do not fall in the first category. Another formation of groups is required for the book value of equity, but this time the companies are divided into three groups rather than two. The stocks are broken in groups based on the book value of equity (BE) to the market value of equity (ME) and the cut-off points are starting from the lowest 30% (S), middle 40% (M), and the highest 30% (H) of stocks in the NYSE. Furthermore, using these groups of stocks, marketable portfolios are constructed each month whose returns are estimated and used for further evaluation.

Step 2: For the second step, the actual indices are formed. The size index, also known as SMB (Small-minus-big) has its value calculated by taking the difference between the simple average of returns on the three small portfolios and the return on the three large portfolios.

The second index known as HML (high-minus-low) has its value estimated by taking the values of monthly returns of high BE/ME portfolio minus the low BE/ME portfolio.

For the purpose of this report, it must be pointed out that in the original paper by Fama and French, the evaluation of the rate of return was calculated using market risk premium which account for the difference in the rate of return of the market and the risk-free rate. To have direct comparison in performance of our models, the equation was modified to only use two extra factors without the market risk premium. The formula used in the report is represented below:

$$R_i = \alpha_i + \beta_1 R_M + \beta_2 SMB + \beta_3 HML + e_i$$

Equation 25 – Fama French 3-factor model

Where:

$\alpha_i - Alpha$

R_i – Expected rate of return

$\beta_i - Sensitivity\ of\ a\ stock\ i\ to\ the\ index$

$R_M - Rate\ of\ market\ return$

$SMB - Small\ minus\ big$

$HML - High\ minus\ low$

$e_i - Random\ term\ independent\ of\ the\ market$

As stated earlier, the Fama-French three factor model represents the expansion of the single-index model introduced in the early 1960s. The additional two factors Fama and French used (SMB and HML) allow the model to be more flexible relative to the single-index model. Furthermore, according to the research findings in literature (Fama, French, 1993) it is found that over the long-term, smaller cap companies will outperform large cap companies and the value stocks will beat growth stocks.

5.2 Application of the Fama-French Three Factor Model

Data of the risk-free rate, SMB and HML that were downloaded directly from the authors website (French and Fama, 2022), the model returns were evaluated. Using the actual and predicted returns, Table 10 was formed where the covariance between the actual and predicted returns is calculated, as well as the standard deviation for both scenarios. Using those results, correlation coefficients were calculated and compared against the Single Index model. It was found that the Fama-French Three factor model outperformed in all 20 stocks, and this can be seen in Figure 10. Furthermore, Figure 11, compares the correlation coefficients of all the previously tested models and it can be seen that the Fama-French Three- factor model, had the highest values of correlation for 10 out of 20 stocks, but it is important to highlight that out of those ten stocks, four come from the Energy sector and four from the financial sector. This highlights the strength of the model in those two sectors, whereas the model had the lowest performance in the Consumer Staples and Communication services.

Table 10 - Performance of the Fama-French Three-factor model

Correlation Results

	KO	WMT	TSN	KMB	ADBE	AAPL
σ_{ij}	0.00096	0.00030	0.00189	0.00037	0.00178	0.00261
σ_i	0.04766	0.05062	0.08581	0.04844	0.06360	0.08196
σ_j	0.03211	0.01826	0.04443	0.02045	0.04277	0.05119
ρ	0.6258	0.3295	0.4949	0.3689	0.6548	0.6225

	MCHP	MSFT	JPM	MCO	ETFC	TRV	CVX
σ_{ij}	0.00451	0.00119	0.00375	0.00307	0.00433	0.00192	0.00262
σ_i	0.09187	0.05055	0.07517	0.06735	0.08585	0.05838	0.06751
σ_j	0.06658	0.03521	0.06249	0.05707	0.06728	0.04575	0.05269
ρ	0.7367	0.6677	0.7982	0.7994	0.7496	0.7173	0.7371

	DVN	MPC	OKE	ATVI	FB	NFLX	OMC
σ_{ij}	0.01308	0.00860	0.00919	0.00079	0.00248	0.00148	0.00129
σ_i	0.1556	0.12331	0.1347	0.08588	0.08098	0.10996	0.06180
σ_j	0.1174	0.09269	0.09851	0.02914	0.04959	0.03646	0.03742
ρ	0.7162	0.7525	0.6923	0.3162	0.6168	0.3692	0.5573

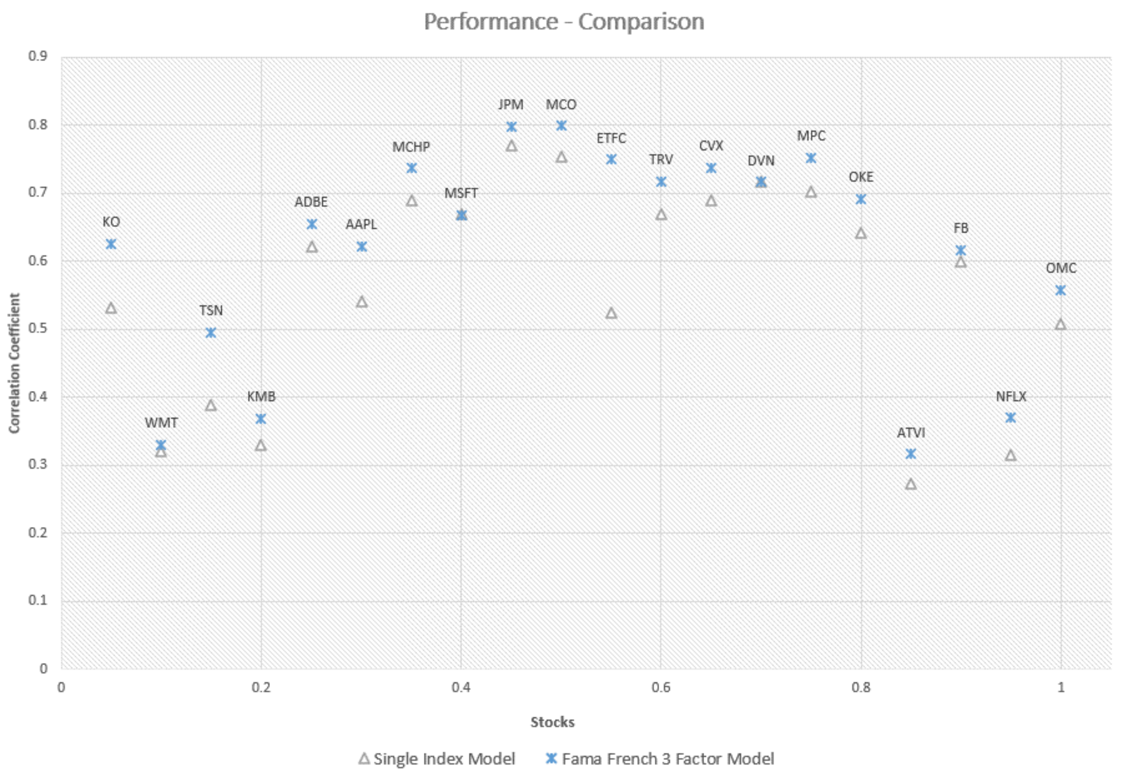

Figure 10 - Comparison between the Single-Index Model and Fama-French Three-Factor Model

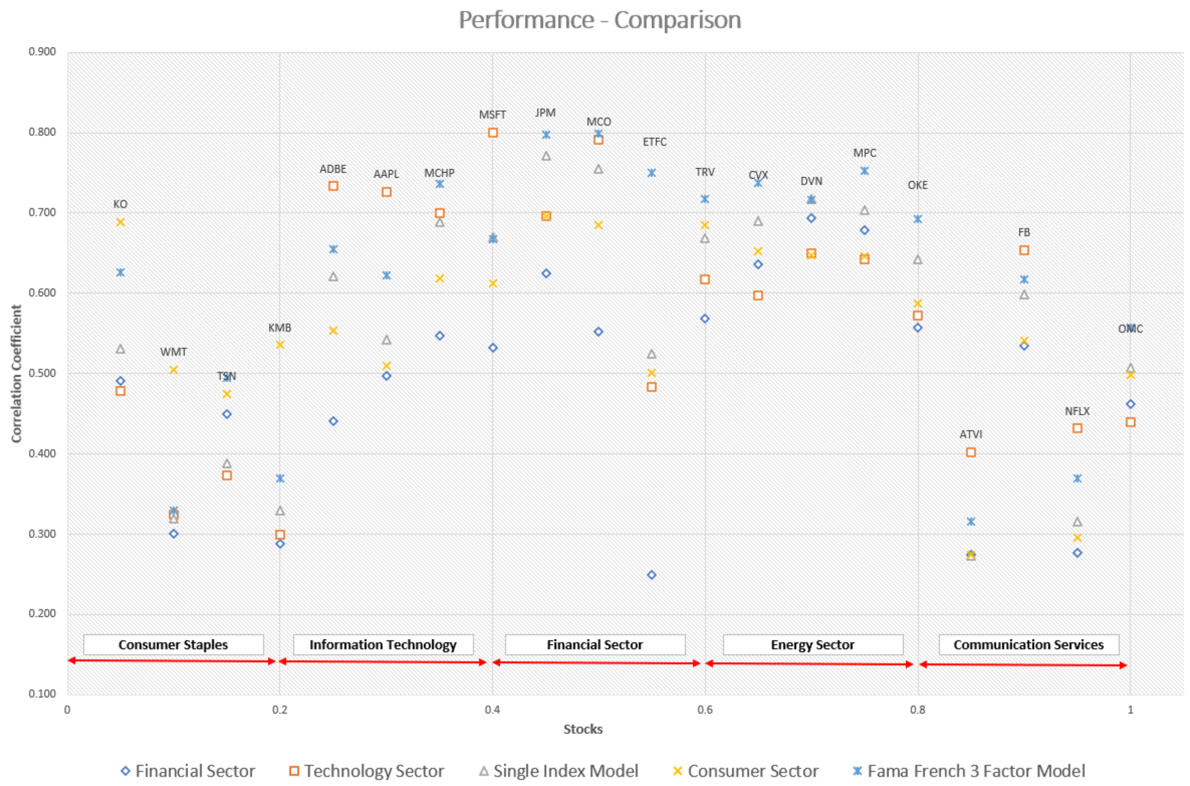

Figure 11 - Comparison of Fama-French Three-factor model with previously tested models

5.3 Fama-French Five Factor Model

The original paper Fama-French (1993) introduced the three-factor model to improve upon the Capital Asset Pricing Model by introducing the size factor and the ratio of the book value of equity to the market value of equity. This allowed for the flexibility and led to increase in predictive performance of the model but since then it was shown that the model with these two factors misses out on a lot of the price movements. (Fama, French, 2014) To account for that, Fama and French introduced a five-factor model in 2014. by adding the factor RMW (robust minus weak) which accounts for the difference between the returns on diversified portfolio of stocks with robust and weak profitability and CMA (conservative minus aggressive), which accounts for the difference between the returns on a diversified portfolio of stocks of low and high investment firms (Fama, French, 2014). The modified equation is as follows:

$$R_i = R_f + \beta_1(R_M - R_f) + \beta_2 SMB + \beta_3 HML + \beta_4 RMW + \beta_5 CMA + e_i$$

Equation 26 – Fama French 3-factor model

Where:

R_i – Expected rate of return

$R_f - Risk - free\ rate$

$\beta_i - Sensitivity\ of\ a\ stock\ i\ to\ the\ index$

$R_M - Rate\ of\ market\ return$

$(R_M - R_f) - Market\ risk\ premium$

$SMB - Small\ minus\ big$

$HML - High\ minus\ low$

$RMW - Robust\ minus\ weak$

$CMA - Conservative\ minus\ aggressive$

$e_i - Random\ term\ independent\ of\ the\ market$

5.4 Application of the Fama-French Five Factor Model

Applying the expanded formula of the Fama-French Three-factor model and adding the robust minus weak and conservative minus aggressive indices, a new set of results is calculated. In general, performance has increased across all stocks, but importantly some significant improvement has been seen in the consumer staples sector, which has quite a poor performance when using only a three-factor model. The performance of the five-factor model is presented in Table 11 and visualized in Figure 12.

Table 11 - Performance of the Fama-French Five factor model

Correlation Results						
	KO	**WMT**	**TSN**	**KMB**	**ADBE**	**AAPL**
σ_{ij}	0.00093	0.00067	0.00190	0.00047	0.00179	0.00263
σ_i	0.04766	0.05062	0.08581	0.04844	0.06360	0.08196
σ_j	0.03080	0.02667	0.04566	0.02274	0.04316	0.05118
ρ	0.6369	0.4937	0.4846	0.4309	0.6518	0.6258

	MCHP	**MSFT**	**JPM**	**MCO**	**ETFC**	**TRV**	**CVX**
σ_{ij}	0.00529	0.00119	0.00384	0.00339	0.00434	0.00197	0.00261
σ_i	0.09187	0.05055	0.07517	0.06735	0.08585	0.05838	0.06751
σ_j	0.07586	0.03474	0.06306	0.06328	0.06697	0.04578	0.05168
ρ	0.7585	0.6768	0.8101	0.7953	0.7554	0.7374	0.7471

	DVN	**MPC**	**OKE**	**ATVI**	**FB**	**NFLX**	**OMC**
σ_{ij}	0.01415	0.00958	0.00956	0.0009	0.0027	0.0015	0.00134
σ_i	0.1556	0.1233	0.1347	0.0859	0.08098	0.0101	0.06180
σ_j	0.1243	0.1003	0.1007	0.0312	0.05496	0.0371	0.03657
ρ	0.7319	0.7748	0.7050	0.3261	0.6122	0.3743	0.5941

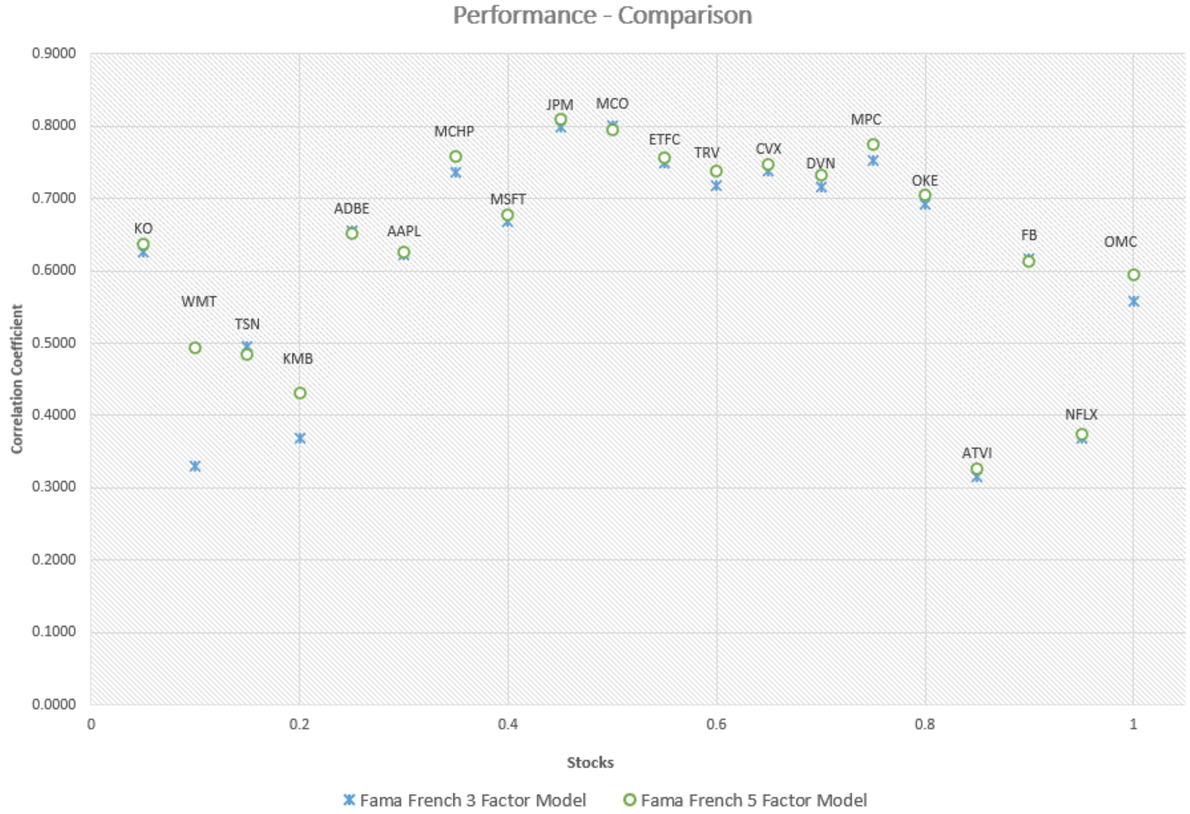

Figure 12 - Comparison in performance between Fama-French Three and Five factor models

5.5 Chen, Roll and Ross

Other than Fama and French, another fundamental multi-index model which laid the foundation for the development of many other models was introduced by Chen, Roll and Ross (1986). Their model is based on two concepts:

1. Value of a share of stock is equal to the present value of future cash flow to the equity holder

Hence, any influence that affects the future cash flow or the function used to evaluate future cash flow can impact the price of a share. So, once the variables that can affect these two parameters are defined, their second concept is formed.

2. Only innovation or unexpected changes to these variables can affect return

Building on the foundation of Chen, Roll and Ross, in a range of articles published by McElroy, Burmeister and several others (Burmeister, Roll and Ross, 2008; & Berry, Burmeister and McElroy, 1988) a common conclusion has been that in order to describe security returns sufficiently, a set of five variables are required. Two variables out of the five focus on the discount rate used to evaluate the present value of the cash flows, the third variable is focused on both discount rates and the size of the cash flow, fourth variable relates only to the cash flow and the final remaining variable is used to capture the impact of the market movement which is not incorporated in the first four variables. This type of model is known as the fundamental risk model and it inspired the creation of the return-generating process developed by Salmon Brothers in 1989, where instead of five variables, their model uses seven variables to explain the return on the securities. (Elton, Gruber, Brown and Goetzmann, 2017)

1. **Economic growth** – To capture the performance of the general economy, the authors argued that the year-to-year changes in the total industry production are a good indicator of the economy health and are much better than the short-term changes due to their fluctuations and the existence of overall long-term bullish trends in the economy.
2. **Business cycle –** The short-term business cycle plays a significant role in the model performance and in order to capture this cyclical behaviour of the economy, the authors take the difference in return on the investment-grade corporate bond and the U.S. Treasury bond.
3. **Long-term interest rate –** Due to the affect of the long-term interest rate has on the attractiveness of the current financial assets, their model uses a yield change in a 10-year Treasury bond.
4. **Short-term interest rate –** The Authors argue that the short-term interest rate in a portfolio of long-term instruments, reduce the available supply of financial assets used for investment. Hence, the model utilizes the yield change in a one-month U.S. Treasury bill.
5. **Inflation –** In order to capture the affect of inflation in the model, the authors use the Consumer Price Index (CPI).
6. **U.S. dollar –** To evaluate the impact the currency fluctuations have on the stock market, the authors argued that a formation of a basket of 15 currencies by trade-weight is required and this yields a statistically stable relationship between their fluctuations and stock returns.
7. Similar to the previous 5-factor model, the last factor in the model is simply the market index uncorrelated with the previous 6 factors.

The formation of this seven-index model had a significant impact on the decision making when selecting indices for a multi-index model creation in this paper. So, even though this paper does not apply this model directly on the benchmark portfolio, it uses some of its variables in the later stages of the report.

6. Chi-square statistical method

Up to this point, the models tested on the benchmark portfolio used correlation coefficient to determine the overall performance of the model. The correlation coefficients utilized covariance of the actual returns and the returns predicted by the model and divided this value with the product of the two standard deviations of actual and predictive returns, respectively. This method, even though it gives a good indication of performance, can be replaced by more advanced statistical methods to verify conclusions found by the correlation coefficients. For the purpose of this paper, the advanced statistical method used is the Chi-square test, introduced by Karl Pearson (1900).

$$\chi^2 = \sum_{i=1}^{n} \left[\frac{(R_{a,t} - R_{p,t})}{\sigma_i} \right]^2$$

Equation 27 – Chi-square test

Where:

$\chi^2 - Chi - square$

$R_{a,t} - Actual\ return\ of\ a\ stock\ at\ the\ value\ of\ t$

$R_{p,t} - Returns\ of\ a\ stock\ predicted\ by\ the\ model\ at\ the\ value\ of\ t$

$\sigma - Standard\ deviation\ of\ the\ actual\ returns\ for\ the\ stock$

6.1 Performance of the models based on Chi square

Table 12 - Chi-square results for the 6 tested models on the benchmark portfolio

	KO	WMT	TSN	KMB	ADBE	AAPL
SIM	30.61	50.24	39.44	43.14	38.98	33.96
IIM_Consumer	22.54	40.27	34.66	30.59	37.70	33.78
IIM_Financial	30.40	49.04	38.02	39.94	34.06	30.04
IIM_Technology	29.73	49.95	38.95	42.70	29.48	16.37
Fama-French 3F	30.17	49.28	36.32	41.79	37.42	34.34
Fama-French 5F	29.63	42.55	37.23	39.15	37.25	34.38

	MCHP	MSFT	JPM	MCO	WFC	TRV	CVX
SIM	23.93	41.92	19.41	23.24	20.32	23.43	22.77
IIM_Consumer	23.43	41.72	17.80	22.65	19.98	22.69	22.61
IIM_Financial	23.87	39.67	5.75	22.46	8.35	22.08	20.97
IIM_Technology	22.39	28.49	16.20	20.72	15.20	22.24	14.13
Fama-French 3F	22.74	42.28	20.35	21.42	20.66	23.33	22.13
Fama-French 5F	21.26	41.42	19.28	21.24	20.21	21.68	21.34

	DVN	MPC	OKE	ATVI	FB	NFLX	OMC
SIM	22.39	20.56	24.57	45.04	31.31	47.31	33.34
IIM_Consumer	20.67	19.26	23.75	44.63	30.68	45.43	32.97
IIM_Financial	20.58	18.52	23.07	35.11	29.67	45.49	28.21
IIM_Technology	19.77	19.70	21.23	34.12	26.06	37.70	27.39
Fama-French 3F	23.07	20.54	25.20	44.67	30.44	46.38	32.52
Fama-French 5F	22.39	18.98	24.27	43.95	30.45	46.21	30.48

Table 13 - Sum of Chi-square test values indicating the best performing model

	$\sum \text{Chi} - \text{Square}$
SIM	635.91
IIM_Consumer	587.81
IIM_Financial	565.30
IIM_Technology	532.52
Fama-French 3F	625.05
Fama-French 5F	603.35

Using the values of the expected return generated by the model and the actual return of the stocks, a chi-square statistical test has been performed to evaluate the performance of the model on stock-by-stock basis and overall portfolio. As seen in the Table 13., the final value of the single-index model performance equals to 635.91 and this gives us the starting value. Any addition of indices to the model should improve the performance, which means that the value of the chi-square will go down. If somehow the value increases, this would be an indication that there is an error in the model. From the results in the same table, it can be concluded that the model with the lowest value of chi-square is the Industry Index Model with the Technology Sector index. This can be reasoned by the fact that even the portfolio is made out of 5 different sectors, each sector has correlated influence of the technology sector, hence the lowest value of chi-square was achieved. Recalling back to Figure 11, where a comparison of the same models based on the correlation coefficients was visualized, it can be concluded that some of the highest values of correlation were achieved by the same model. The superior performance of the Industry Index Model with a technology sector index was then confirmed with the chi-square test, but this doesn't overall indicate that this model would work the best in predicting the stock market returns as the model utilizes two factors: market performance and technology sector index performance. Observing the S&P500 index, some of the largest market cap companies are in the technology sector, so the superior performance could come from the fact that market index is dominated by the technology companies. Another interesting fact found in the results was the relatively poor performance of the Fama-French three and five factor models. Nevertheless, using sector indices as multi-index model indices might not be the best for building a predictive model, in order to further evaluate the performance of Fama-French model, a separate study will be conducted later in the report.

7. Building a multi-index model

Up to this point, this report has been focusing on the analysis of a small batch of 20 stocks which were used initially to test the performance of the single-index, industry-index, and Fama-French models. From the results shown in previous section it can be concluded that there are some conflicting results due to the superior performance of the Industry Index model with technology sector. This may be due to the fact that the companies in the benchmark portfolio that belong to the technology sector have a higher market cap compared to the consumer and energy sector. As a result of that, there is an imbalance, and the performance of models is questionable. To resolve this issue, for the purpose of building and testing a multi-index model, a random sample of 250 stocks from the S&P500 have been generated and will be applied in the future models. The testing period selected for the multi-index models is set from the year 2015 to 2018 for a total of 36 trading months.

7.1 Indices selection

Based on the work of Burmeister, Roll and Ross (2008); Berry, Burmeister and McElroy, (1988) and Fama and French(1992), a total of 14 indices were selected for building a multi-index model. The indices included in the models are:

1. Interest rate of 30 Year U.S. Treasury Bond
2. Interest rate of 10 Year U.S. Treasury Bond
3. Interest rate of 5 Year U.S. Treasury Note
4. Interest rate of 13-Week U.S. Treasury Bill
5. Price of Gold
6. Price of Oil
7. Price of Silver
8. Value of U.S. Dollar
9. Inflation (CPI - Consumer Price Index)
10. SMB (small minus big)
11. HML (high minus low)
12. CMA (conservative minus aggressive)
13. RMW (robust minus weak)
14. Market index uncorrelated with all the previous indices

There is a fundamental reason why the first four selected indices are treasury bonds, notes and bills. During the economic expansion, the relationship between the bond prices and stock prices are inversely proportional, because the belief in the stock market is high, the investors tend to move the money from the bond market to the stock market. This is because investors expect higher yield in the stock market, so the bond prices go down, but the yield of the bonds tends to go up to attract investors back in the bond market. During an economic contraction, the investors fear for their money, so they move it into the bond market, which increase the overall demand for the bonds, and this drives the prices of the bonds up and the yields down. This is fundamentally true, because the default risk of the U.S. government is very low, compared to publicly traded companies. (Pilotte and Sterbenz, 2006; Plackett, 1983; Shen, 1995)

When observing the prices of gold, their fundamental behaviour is driven by several factors, including the supply of the gold which is relatively fixed on year-to-year basis, demand for the gold and the investors behaviour. According to the Hood and Malik (2013), gold can be seen as a hedge (negatively correlated with stocks) and a safe haven (negatively correlated with stocks in extreme stock market events) by studying the gold prices from 1995 to 2010. On the other hand, authors like Claude B. Erb & Campbell R. Harvey (2013) have found in their studies that gold doesn't correlate strongly with inflation, which means that when inflation rises, it does not necessarily mean that gold is a good bet against it. From a fundamental standpoint, as inflation rises, the paper money loses value as more of it is being printed, so one could assume that gold would be a good bet, but this was found not to be the case, so the real question is "What is the real reason behind the price of gold?" and even more importantly, "How does the gold price, affect the performance of the stock market?". Due to these questions, gold prices have been selected as part of the multi-index model studied in this paper.

Another important global commodity that can have a huge impact on the economy is crude oil. There are many investigations on whether direct impact of the oil prices on the stock market performance and according to Kilian and Park (2009), the reaction of the U.S. stock market to changes in the oil prices differ based on if the price changes occurred due to demand or supply shocks in the oil market. Another study done by Perry Sadorsky (1999) found through vector autoregression, that oil prices and oil price volatility play significant role in the real stock returns. In addition to that, it was found that after 1986, the oil price movements explain a larger fraction of the forecast error variance in real stock returns than interest rates do. Observing the matter sensibly, the reduction in the price of oil mean lower prices of transportation which leads to higher disposable income in people's possession that can go in the economy. As well as that, the oil is not used only for transportation, but many industrial chemicals and processes depend on oil, hence this reduction in oil prices would lead to lower manufacturing cost and a more thriving business. Nevertheless, there are sectors which depend heavily on the oil prices, such as transportation but with the development of green energy initiatives in the past decade, this dependence on the oil is being reduced and so is the impact on the stock market return. Conclusively, this global commodity was selected as part of the multi-index model formation discussed in this paper.

The next commodity that investors turn to in various economic crises alongside gold, is silver and in case of silver, which has a really high industrial use (about 56% of total supply), investors believe the precious metals will hold in value, despite the market performance. Observing the historical performance of gold and silver during the stock market crashes in Figure 13., we can see confirm our previous statements on gold that investors turn to gold due to the fear of their money becoming worthless, hence there is a negative correlation between the gold prices and S&P500. Only during the crash of 1980, also known as "Silver Thursday", did gold and silver prices crash more than the market itself, but generally, gold prices tend to rise during the market crash. On the other hand, observing the silver prices, due to their high industrial use, their prices trace market performance more closely than gold. Nevertheless, the average crash of the market is -30.9%, whereas the average dip of the silver prices is -16.4%. It was also found that the silver would have smaller drops in prices if the crash occurred after a bull run, but if it did not, the silver would follow the market with greater correlation. (Sridhar, Sumathy, Sudha and Ambrose, 2016). Another significant impact that occurs related to the changes in the silver prices is the impact on the solar energy firms because silver is heavily utilized in photovoltaic (PV) processes for producing solar energy. (Dutta, 2019)

Gold Performance During Stock Market Crashes GoldSilver

Dates of S&P 500's Biggest Declines	S&P 500	Gold	Silver
Sep 21, 1976 - Mar 6, 1978	-19.4%	53.8%	15.2%
Nov 28, 1980 - Aug 12, 1982	-27.1%	-46.0%	-66.1%
Aug 25, 1987 - Dec 4, 1987	-33.5%	6.2%	-11.8%
Jul 16, 1990 - Oct 11, 1990	-19.9%	6.8%	-10.8%
Jul 17, 1998 - Aug 31, 1998	-19.3%	-5.0%	-9.5%
Mar 27, 2000 - Oct 9, 2002	-49.0%	12.4%	-14.4%
Oct 9, 2007 - Mar 9, 2009	-56.8%	25.5%	1.1%
May 10, 2011 - Oct 3, 2011	-19.0%	9.4%	-19.1%
Feb 19, 2020 - Mar 23, 2020	-33.9%	-4.9%	-31.8%
AVERAGE	**-30.9%**	**6.5%**	**-16.4%**

Table 14 - Gold and Silver performance during the Stock Market Crashes (The Effect of a Stock Market Collapse on Silver & Gold, n.d.)

Following the commodities like gold, silver and oil, the next index included in the model analysis is the U.S. Dollar. There are only two ways that the value of U.S. Dollar, or any country's currency can be increased and that is through the global demand of that currency or if the central bank reduces the amount of the currency available. There are several possibilities how the value of U.S. dollar can impact the return of the stocks in the portfolio. Firstly, if the companies in the portfolio are heavily depended on the import of raw materials to produce the final product, with the decline of the U.S. dollar value, there is a decline in the purchasing power. This means, that it will be more expensive to purchase the raw materials which directly impacts the company. If the company decides to transfer that cost to their customers, this can lead to the potential loss of customers, or the company can decide to absorb that cost, but it will lead to smaller profit margins and impact the bottom line. Another possibility is that the companies that focus on the export of U.S. manufacturing goods will do better if the dollar declines because they will get more dollars when the convert the foreign cash that was introduced to the company by overseas buyers. In the study of the relation of the US dollar to the stock market, Samith Azar (2015) found that there is a strong correlation between the two and this was also confirmed by Robert Johnson and Luc Soenen (2004) in their research "The US stock market and the international value of the US dollar".

The remaining indices tested in the model come from the work of Fama and French (1993) and they are: SMB (small minus big), HML (high minus low), CMA (conservative minus aggressive) and RMW (robust minus weak). The SMB will help to explain the effect of the company size on its return, the HML focuses on the ratio of the book market of equity to the market value of equity, the RMW helps explaining the impact of the company's profitability and CMA accounts for the difference between the returns on a diversified portfolio of stocks of low and high investment firms. The last index is the market index which will explain all of the remaining price movements which were not captured by the previously stated indices.

7.2 Formation of models and results

Before introducing the formation of the multi-index model, it is important to mention that the entire process of building the optimal multi-index model required the creation of 91 multi-index models, before the final selection takes place. The process started with the creation of the Single-Index Model, where the underlying assumption is that the only reason stocks move together, systematically, is because of a common co-movement with the market and that there are no other effects beyond it such as industry or sector effects. Observing the Equation 28., which provides the return on the stock i based on a multi-index model that has n number of indices and n number of betas, which allows for the expansion of the single-index model by adding the new index I_{n+1} which must be uncorrelated with all of the previous indices in the model and $\beta_{i(n+1)}$ which provides the sensitivity of the stock return to the new index.

$$R_i = a_i + \beta_{i1}I_1 + \beta_{i2}I_2 + \cdots + \beta_{in}I_n + c_i$$

Equation 28 - Return on the security i based on the multi-index model with n number of indices that are uncorrelated with one another

This technique was applied to the created single-index model to expand it into a total of 13 two-index models, where each model had the market factor, plus one of the 13 indices chosen for the study which were introduced in the previous sub-section. The evaluation process was based on calculating the model returns and running the chi-square statistical method to determine the performance of each model. The results can be seen in the Table 14. below.

Table 15 - Chi-square for SIM and 13 two-index models

	Chi-square
SIM	6588.924
SIM + SMB	6416.67
SIM + HML	6376.561
SIM + RMW	6405.442
SIM + CMA	6381.995
SIM + CPI	6402.764
SIM + 13w T. Bill	6349.132
SIM + 5y T. Note	5817.939
SIM + 10y T. Bond	5802.301
SIM + 30y T. Bond	5900.912
SIM + Oil	6242.412
Sim + Gold	6155.586
SIM + Silver	6217.585
SIM + US Dollar	6363.386

The firstly created Single-index model, yielded a value of chi-square of 6588.924 which set the performance base-line. With the addition of each factor, we can evaluate their effect on the stock price movements, based on the amount the chi-square was reduced. As 13 two-index models were created where the only differentiation was the second index used, a direct influence of the index can be observed. From the Table 14., it is clear that the best performing model was a two-index model, which used a market factor and a 10-year U.S. Treasury Bond. Using these results, a second stage of analysis could be conducted where a new base-line represents the winning two-factor model “SIM + 10y T. Bond” and the next step was to create 12 three-index models, where each model will have the market factor, the 10-year U.S. Treasury Bond and one extra index from the list of indices selected for the analysis.

Observing the results from the Table 15., we can see that the base-line two-index model has a chi-square value of 5802.3. Applying the same principles of the chi-square reduction, the third index which reduced the chi-square the most was the price changes of gold. These results contradict the findings of Claude B. Erb (2013) who stated that gold prices have no correlation with the stock-market performance, but our findings do confirm the results found by Matthew Hood (2013). The winning three-factor model yielded a final chi-square value of 5520.95.

Table 16 - Chi-square for the winning 2-Factor Model and 12 three-index models
2FM – Winning two-factor model from the previous stage (SIM + 10y T. Bond)

	Chi-square
2FM	5802.30
2FM + SMB	5632.72
2FM + HML	5640.66
2FM + RMW	5605.78
2FM + CMA	5578.77
2FM + CPI	5659.84
2FM + 13w T. Bill	5592.95
2FM + 5y T. Note	5640.50
2FM + 30y T. Bond	5559.00
2FM + Oil	5544.56
2FM + Gold	5520.95
2FM + Silver	5589.38
2FM + US Dollar	5575.56

Table 17 - Chi-squared for the winning 3-Factor Model and 11 four-index models
3FM – Winning three-factor model from the previous stage (SIM + 10y T. Bond + Gold)

	Chi-square
3FM	5520.95
3FM + SMB	5362.30
3FM + HML	5359.70
3FM + RMW	5334.43
3FM + CMA	5312.13
3FM + CPI	5376.73
3FM + 13w T. Bill	5366.10
3FM + 5y T. Note	5368.27
3FM + 30y T. Bond	5289.31
3FM + Oil	5307.00
3FM + Silver	5367.75
3FM + US Dollar	5332.50

Continuing on with the process, the four-factor model was built upon the winning three-factor model which consisted of the market index, 10 Year US Treasury Bond and Gold. This three-index model has seen a reduction of chi-square from 5802.3 to 5520.95. Comparing this to the performance of the four-factor model where from the Table 16. we can see that the index which reduced the chi-square of the three-factor model the most was the 30 Year US Treasury Bond, which yielded a final value of 5289.31 for the chi-square of the four-factor model. This confirms the findings in (Pilotte E., 2006 and Shen P., 1995) and we can see that there is a high correlation between the stock market and the bond market.

Next stage required the analysis of 10 five-index models, where the base-line performance was set by the winning 4 factor model from the previous stage that yielded a value of chi-square of 5289.31. In this stage, two factors reduced the chi-square the most and that was the CMA factor from Fama-French whose model yielded a value of 5079.38 and the price changes of oil whose model yielded a chi-square of 5062.21 which can be seen in Table 17. Hence the winning five-factor model taken for further analysis contains the market factor, 10- and 30-Year Treasury Bond, Gold and Oil.

Table 18 - Chi-square for the winning 4-Factor Model and 10 five-index models

4FM – Winning four-factor model from the previous stage (SIM + 10y T. Bond + Gold + 30y T. Bond)

	Chi-square
4FM	5289.31
4FM + SMB	5118.95
4FM + HML	5128.75
4FM + RMW	5099.30
4FM + CMA	5079.38
4FM + CPI	5138.59
4FM + 13w T. Bill	5134.35
4FM + 5y T. Note	5187.60
4FM + Oil	5062.21
4FM + Silver	5135.42
4FM + US Dollar	5119.37

Table 19 - Chi-square for the winning 5-Factor Model and 9 six-index models

5FM – Winning five-factor model from the previous stage (SIM + 10y T. Bond + Gold + 30y T. Bond + Oil)

	Chi-square
5FM	5062.21
5FM + SMB	4895.17
5FM + HML	4903.13
5FM + RMW	4879.63
5FM + CMA	4862.34
5FM + CPI	4906.46
5FM + 13w T. Bill	4913.46
5FM + 5y T. Note	4959.32
5FM + Silver	4912.90
5FM + US Dollar	4895.31

Following that, the formation of six-factor model resulted in the Fama-French CMA (Conservative minus aggressive) to be the winning factor. The model with CMA yielded a final value of the chi-square of 4862.34 compared to the base 5-factor model that had a value of 5062.21 which can be seen in Table 18.

Table 20 - Chi-square for the winning 6-Factor Model and 8 seven-index models

6FM – Winning six-factor model from the previous stage (SIM + 10y T. Bond + Gold + 30y T. Bond + Oil + CMA)

	Chi-square
6FM	4862.34
6FM + SMB	4725.25
6FM + HML	4711.91
6FM + RMW	4678.47
6FM + CPI	4705.06
6FM + 13w T. Bill	4713.84
6FM + 5y T. Note	4760.14
6FM + Silver	4713.87
6FM + US Dollar	4694.21

Following the CMA (Conservative minus aggressive) index from Fama-French, the next stage showed that another Fama-French index reduced the chi-square value the most and that was the RMW (Robust minus weak). The final 7-factor model which included the RMW reduced the chi-square value to 4678.4 which can be seen in Table 19.

The process was continued until all of the indices were used (Tables 20-26). After the Fama-French factors, the best performing indices are US Dollar, Silver, HML, 13 Week Treasury Bill, SMB, 5 Year Treasury Note and CPI, consequently.

Table 21 - Chi-square for the winning 7-Factor Model and 7 eight-index models

7FM – Winning seven-factor model from the previous stage (SIM + 10y T. Bond + Gold + 30y T. Bond + Oil + CMA + RMW)

	Chi-square
7FM	4678.47
7FM + SMB	4528.25
7FM + HML	4524.23
7FM + CPI	4523.04
7FM + 13w T. Bill	4530.87
7FM + 5y T. Note	4568.88
7FM + Silver	4529.66
7FM + US Dollar	4515.10

Table 22 - Chi-square for the winning 8-Factor Model and 6 nine-index models

8FM – Winning eight-factor model from the previous stage (SIM + 10y T. Bond + Gold + 30y T. Bond + Oil + CMA + RMW + US Dollar)

	Chi-square
8FM	4515.10
8FM + SMB	4362.16
8FM + HML	4363.90
8FM + CPI	4375.06
8FM + 13w T. Bill	4366.99
8FM + 5y T. Note	4405.88
8FM + Silver	4356.08

Table 23 - Chi-square for the winning 9-Factor Model and 5 ten-index models

9FM – Winning nine-factor model from the previous stage (SIM + 10y T. Bond + Gold + 30y T. Bond + Oil + CMA + RMW + US Dollar + Silver)

	Chi-square
9FM	4356.08
9FM + SMB	4198.10
9FM + HML	4157.44
9FM + CPI	4217.79
9FM + 13w T. Bill	4196.75
9FM + 5y T. Note	4232.72

Table 24 - Chi-square for the winning 10-Factor Model and 4 eleven-index models

10FM – Winning ten-factor model from the previous stage (SIM + 10y T. Bond + Gold + 30y T. Bond + Oil + CMA + RMW + US Dollar + Silver + HML)

	Chi-square
10FM	4157.43
10FM + SMB	3999.89
10FM + CPI	4013.69
10FM + 13w T. Bill	3995.73
10FM + 5y T. Note	4017.99

Table 25 - Chi-square for the winning 11-Factor Model and 3 twelve-index models

11FM – Winning eleven-factor model from the previous stage (SIM + 10y T. Bond + Gold + 30y T. Bond + Oil + CMA + RMW + US Dollar + Silver + HML + 13w T. Bill)

	Chi-square
11FM	3995.73
11FM + SMB	3840.24
11FM + CPI	3848.03
11FM + 5y T. Note	3847.14

Table 26 - Chi-square for the winning 12-Factor Model and 2 thirteen-index models

12FM – Winning twelve-factor model from the previous stage (SIM + 10y T. Bond + Gold + 30y T. Bond + Oil + CMA + RMW + US Dollar + Silver + HML + 13w T. Bill + SMB)

	Chi-square
12FM	3840.24
12FM + CPI	3692.39
12FM + 5y T. Note	3688.01

Table 27 - Chi-square for the winning 13-Factor Model and 1 fourteen-index model

13FM – Winning thirteen-factor model from the previous stage (SIM + 10y T. Bond + Gold + 30y T.Bond + Oil + CMA + RMW + US Dollar + Silver + HML + 13w T. Bill + SMB + 5y T. Note)

	Chi-square
13FM	3688.019
13FM + CPI	3527.207

A summary table of all of the selected models with their Chi-square values is shown below.

Table 28 - Summary table

Model	Chi-square Sum
1 Factor Model	6588.92
2 Factor Model	5802.30
3 Factor Model	5520.95
4 Factor Model	5289.31
5 Factor Model	5062.22
6 Factor Model	4862.34
7 Factor Model	4678.48
8 Factor Model	4515.11
9 Factor Model	4356.08
10 Factor Model	4157.44
11 Factor Model	3995.73
12 Factor Model	3840.25
13 Factor Model	3688.02
14 Factor Model	3527.21

7.3 Selection using an information criterion

Observing the results in the previous section, it can be concluded that as more indices were added to our models, the chi square value was being reduced, which indicates the increase in the performance of the multi-index model. Technically, this is correct, but it doesn't take into consideration the complexity that each index introduces to the equation. The most important question when formulating the optimum multi-index model is "where is the cut-off point in the indices selection?". Theoretically, by introducing every available index, it would be possible to create a better and better representation of the data, but too much complexity will weaken the predictive power of the model. Thereby, there must be a trade-off between the goodness of fit of the model and the simplicity of the model. Statistical information criteria are used to evaluate this trade off. The one applied in this report is the Akaike information criterion (AIC) (Burnham and Anderson, 2004). AIC is based on the information theory by a Japanese statistician Hirotugu Akaike, and it is used as an estimator of prediction error, which is essentially the relative quality of the statistical models for a given set of data. From the previous section it was seen that we were evaluating the performance from the single-index model all the way up to the fourteen-factor model. The AIC takes the quality of each of the models relative to each other and with this it provides the means for model selection. (Burnham K., 2004)

$$AIC = N\,ln\left(\frac{\chi^2}{N}\right) + 2K$$

Where:

$\chi^2 - Chi - square$

$N - Number\ of\ observations$

$K - Number\ of\ parameters\ used\ in\ the\ model + 1$

Interpreting the data from the Table 29, it can be seen that the starting value of the AIC for the single-index model was 191.55. Keeping this in mind, the approach applied here would be to find a model with the maximum reduction in the chi-square value to increase the quality of the model but to keep the value of AIC below that of the single-index model. Observing the Seven Factor Model, the AIC value is 191.22 which is lower than the single-index value of 191.55 but the chi-square value was reduced from 6588.92 to 4678.48. This provides us with the maximum trade-off between the goodness of fit and the quality of the model. If complexity is taken as the primary concern in the model selection, then the obvious model would be the Two Factor Model because it has the lowest value of the AIC. However, the selection process in this report focuses on the maximum trade-off between the goodness of fit and the quality of the model. By selecting the seven Factor Model, we get the highest quality model with respect to its predictiveness.

Table 29 - AIC results for generated multi-index models

Model	Chi-square Sum	K	N	AIC
1 Factor Model	6588.92	2	36	191.55
2 Factor Model	5802.30	3	36	188.97
3 Factor Model	5520.95	4	36	189.18
4 Factor Model	5289.31	5	36	189.64
5 Factor Model	5062.22	6	36	190.06
6 Factor Model	4862.34	7	36	190.61
7 Factor Model	4678.48	8	36	191.22
8 Factor Model	4515.11	9	36	191.94
9 Factor Model	4356.08	10	36	192.65
10 Factor Model	4157.44	11	36	192.97
11 Factor Model	3995.73	12	36	193.54
12 Factor Model	3840.25	13	36	194.11
13 Factor Model	3688.02	14	36	194.66
14 Factor Model	3527.21	15	36	195.05

7.3.1 Final Selection

Based on the chi-square statistical method and the Akaike Information criterion discussed in the previous subsection, it was concluded that the optimum multi-index model was a seven-index model, whose factors are:

- Market factor,
- 10 Year Treasury Bond,
- Gold,
- 30 Year Treasury Bond,
- Oil,
- CMA and
- RMW

In order to test the performance and validate our multi-index model, two additional periods were selected was the year 2019 and the period of extreme events 2020-2021. The idea behind testing our model in the years 2020-2021 is to see how the model performs during the unusual market conditions caused by the start of the pandemic.

7.4 Application of the model to 2019

In the interest of validating the results obtained by the Akaike information criterion and to evaluate the overall performance of our multi-index models, it was required to test the performance and the predictiveness of the model in a different year. As the original training period of the 250 stocks portfolio, was the period from 2015 to 2018 for the total of 36 trading months, the idea here was to test how well does the model perform in the year 2019 for a total of 12 trading months. To achieve this, the betas of the seven-factor model were extracted from the original model and applied to the first seven factor model in the year 2019. The reasoning behind this is that if the model in the original testing period found that the company Adobe Inc. has a 30 Year U.S. Treasury Bond beta value of 0.75, which means that for each 1% that the 30 Year U.S. Treasury Bond goes up/down, the model is saying that the Adobe Inc. would move 0.75% up/down. This allows for the predictiveness of the model, because if we know the relationship between a stock and an index, we can project this relationship in the future and estimate the performance of the stock.

To be able to compare the performances of the models in the two periods, one covering 36 months and one 12 months. The chi-square will be normalised by dividing the value of chi-square with the number of months.

Table 30 - Chi-square standardized for 2015-2018 and 2019 testing periods

Model	Chi-square Normalized	
	2015-2018	2019
1 Factor Model	183.03	156.14
2 Factor Model	161.18	155.98
3 Factor Model	153.36	155.96
4 Factor Model	146.93	157.41
5 Factor Model	140.62	158.25
6 Factor Model	135.07	160.04
7 Factor Model	129.96	159.53

Reflecting on the Table 28, it can be observed that in the 2019 testing year, the chi-square value starts at 156.14. It begins to decrease to the value of 155.96 for the three-factor model, following which it begins to increase again. This is a very interesting finding because it indicates that the model with the best predictiveness is the three-factor model, rather than the seven-factor model implied by the Akaike information criterion. Besides, this goes to add that the penalty in the Akaike information criterion for adding extra indices in the model is more lenient than it should be. Nevertheless, the chi-square value of for the three-factor model in 2019., does not increase drastically in the seven-factor model, as the matter of fact, the value of chi-square even decreases going from the six-factor to the seven-factor model from the value of 160.04 to 159.53. So, the final conclusion here is that our model was not far off in the estimation of the best trade-off between the goodness of fit and the simplicity of the model by indicating that the seven-factor model is the best. However, our test indicates, that the three factor model works best when applied to another period.

7.5 Application of the model to 2020-2021

Following the discussion in the previous section, the model performance was also tested for the years 2020-2021 for a total of 24 trading months. This was the period in which the pandemic started, and unusual market activity occurred. It is beneficial to understand how our model behaves in the time of uncertainty and extreme market events. I followed the same procedure when testing the models for 2019, except that the chi-square was normalised by dividing by 24 and the results are as follows:

Table 31 - Chi-square standardized for 2015-2018, 2019 and 2020-2021 testing periods

Model	Chi-square Normalized		
	2015-2018	2019	2020-2021
1 Factor Model	183.03	156.14	162.24
2 Factor Model	161.18	155.98	172.77
3 Factor Model	153.36	155.96	176.09
4 Factor Model	146.93	157.41	186.62
5 Factor Model	140.62	158.25	193.66
6 Factor Model	135.07	160.04	198.70
7 Factor Model	129.96	159.53	202.68

Comparing the results from the 2020-2021 to the ones for year 2019, it can be seen that they do not follow the same pattern. The normalised chi-square for 2020-2021 starts rising immediately after adding the second factor to the single-index model and the rise in values is also much greater compared to the one in 2019. The model performance in the case of unusual market activity caused by the pandemic seems to be invalidated and points out that even the three-factor model working well for the 2019, performed worse than the single-index model. Assumption for this behaviour is that when extreme market events occur, the stocks and the market tend to be heavily correlated, so the addition of any extra indices can only input extra noise and overfit the model.

8. Discussion

The initial idea behind this project was to investigate and build quality multi-index models which will be used for reproducing the historical prices of assets and predict their future performance. To begin with, a construction of a benchmark portfolio consisting of 20 stocks was conducted where the stocks were chosen from five different sectors in the S&P500 and four stocks from each sector were chosen. Two of the stocks were from the higher market cap range and two were from the lower market cap range to balance the portfolio. The introductory analysis was focused on the single index model whose underlying assumption is that the rise or fall of asset prices is explained by the changes in the market. This model, which was originally first explored by William Sharpe in 1963, was to an extent validated by the first results in this paper where a correlation coefficient for some stocks explained as much as 77% of the rises and falls in a four-year period, which is very promising. Nevertheless, the correlation coefficient on some stocks was as low as 27% which is not as efficient as expected.

There were two fundamental issues with this approach, the first one is that the assumption in the single-index model which states that the only reason stocks move together, systematically, is because of a common co-movement with the market is insufficient. There are numerous economic and non-economic factors which influence the prices of stocks, but the single-index model approach provides a good starting point for development into a multi-index model analysis. The second issue is the evaluation of performance of the model by utilizing the correlation coefficient which uses the covariance between the actual returns and returns predicted by the model with the standard deviation of each. The fact is that there are several statistical methods which can provide us with the higher quality analysis and give insight into the quality of the model performance.

The solution to the first problem was done by the extension of the single-index model into the sector-index model, which is a simple two-factor model that gives insight into the co-movement of stock prices with the market as well as the sector. It must be noted that correlation between market index and sector indices were removed. In this report, three sector-index models were tested which included the analysis of Financial Select Sector (XLF), Consumer Staples Select Sector (XLP), and Technology Select Sector (XLK). By introducing the additional indices in the single-index model, it was possible to capture extra movements of the stock prices. This was arguably visible for stocks from the financial sector, when using the sector-index model with the financial sector index, the technology companies when using the sector index model with the technology index and so on. An example of the increase in performance can be seen when observing the correlation coefficient of Walmart

(WMT) in the single-index model which explained only 32% of the price movement and this value jumped to 50% when using a consumer sector-index model.

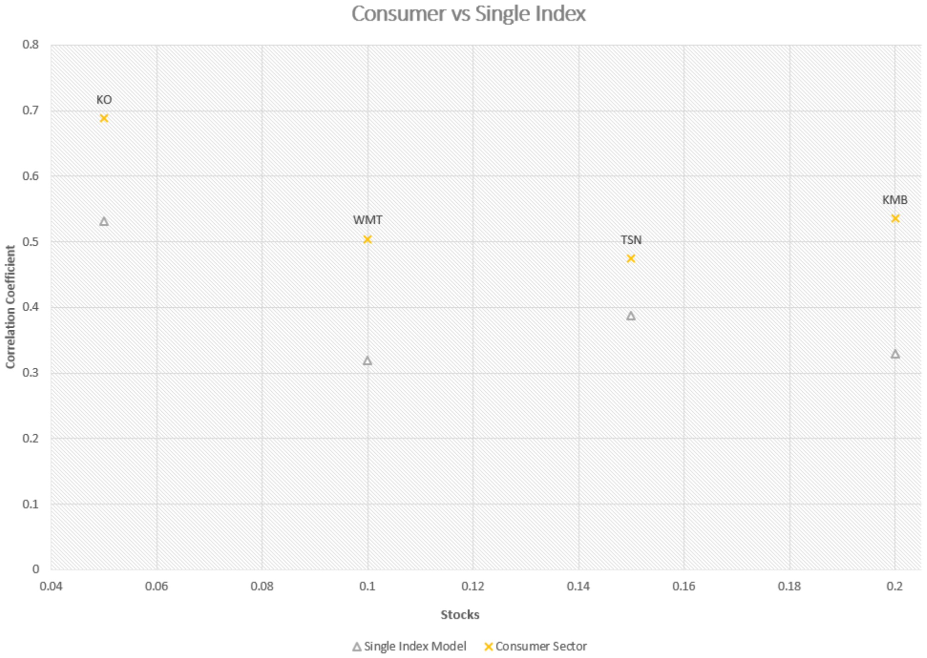

Figure 13 - Correlation coefficient of Consumer vs Single Index Model (Repeat of Fig. 7)

In theory, these results are promising but observing the problem fundamentally, it must be noted that using a sector performance to explain the price movements of stocks, can be misleading, because the sector index is formed by grouping of the companies in that sector, similarly to the creation of the market index. This brings to question; can some indices just reflect the movement of a few dominant stocks?

In case of diversification, it is better to explore some other economic factors. Before divining into the exploration of the economic factors that can be used to explain stock price movements, a study of some of the first fundamental multi-index model is carried out. The main two fundamental multi-index models reviewed were Fama and French Three Factor and Five Factor Models. Building on the Single-Index model, Fama and French argued that both the market cap (company size) and the ratio of the book value of equity to the market value of equity have a strong impact on the average return of common stocks. This led to the creation of two extra indices added to the single-index model: small-minus-big (SMB) and high-minus-low (HML). Using the data for SMB and HML from the authors website, a Fama French Three factor model was constructed in a way where the two indices were uncorrelated from the market and from one another and added to the single-index model, expanding it into a three-factor model. After obtaining the correlation coefficients on stock-by-stock basis, it could have be seen that the newly formed three factor model outperformed the single-index model for every stock but comparing it to the previously tested sector-index models, it only outperformed for 10 out of 20 stocks. Out of those 10 stocks, 8 stocks are from the energy and financial sector which indicates the strength of the model in those sectors. The reason for the lower performance could be coming from the fact that only a small sample of stocks was chosen in the benchmark portfolio, so the performance of the sector-index models was influenced by the high market cap companies. Nevertheless, it was found that the three-factor model was missing out on a lot of price movements which was also stated by Fama and French who for that reason expended their original three factor model by introducing the RMW (robust minus weak) index which accounts for the difference in company profitability and CMA (conservative minus aggressive) which accounts for difference in returns of the stocks of low and high investment firms. Applying the five-factor model, it was confirmed that the overall performance of the model has increased across all stocks and more price movement has been explained as expected.

Up to this point, the results discussed were all evaluating correlation coefficient. More advanced statistical methods exist that can provide a more accurate insight into the overall performance of the models. In this report the chi-square statistical test is applied. Running the test on the previously discussed models, the first step was to get a benchmark value of chi-square for the single-index model as everything was built on top of it. The chi-square value for the single-index model was 635.91 and now it was expected from each model to reduce this value leading to a higher quality model. Not surprisingly, the highest performing model was the expanded single index model with the technology sector index, and this was confirmed both by the chi-square test (value of 532.52) and the correlation coefficients. This can be reasoned with the fact that this model was utilizing two factors: market performance and technology sector performance. Observing the S&P500, it can be seen that most of the highest market cap companies are from the technology sector, so their movement contribute the highest to the entire movement of the market, hence, the two-factor model with technology sector index provides the most explanation to the stock price movements. On the other hand, the unexpected results came from the performance of the Fama-French Three and Five factor models, where the three-factor model reduced the chi-square value to 625.05 and the five-factor model to 603.35. Even though, this doesn't imply that the Fama-French models are not good, it could be from the fact that only 20 stocks and three sectors were used in the benchmark portfolio, so an obvious step for further analysis was to increase the sample size which led to the creation of a new portfolio of 250 stocks.

Table 32 - Sum of Chi-square test values indicating the best performing model (Repeat of Table 13.)

	$\sum \mathbf{Chi-Square}$
SIM	635.91
IIM_Consumer	587.81
IIM_Financial	565.30

IIM_Technology	532.52
Fama-French 3F	625.05
Fama-French 5F	603.35

With this new portfolio of 250 stocks and a chi-square statistical method for testing the performance of indices in the model, it was now required to select the indices for testing. Following steps included the comprehensive literature review of the existing multi-index models (Chen, Roll, and Ross, Burmeister and McElroy, Fama and French, Claude B. Erb, Lutz Kilian, Perry Sadorsky). A total of 14 indices have been selected for building the multi-index models. These were constructed for the years 2015-2018 with a total of 36 months (the training period). The indices include 10- and 30-Year U.S. Treasury Bond, 5 Year U.S. Treasury Note, 13 Week U.S. Treasury Bill, Gold, Oil, Silver, U.S. Dollar, Inflation, SMB, HML, CMA, RMW and a market index uncorrelated with all the previous indices. The process started off by running a single-index model on the new 250 stocks portfolio and using the multi-index model equations, 13 two-factor models were created where each model had the market index plus one of the other indices selected. Running the chi-square test, it was found that the starting value for the single-index model was 6588.92 and the first index that was selected for building the optimum multi-index model was the 10 Year U.S. Treasury Bond as the two-factor model containing this index and market index, reduced the original chi-square value the most, to a value of 5802.301. This two-factor model now became a new benchmark value, and another 12 three-index models were created where the model which reduced the chi-square value the most was selected, and the process continued on until there were no more indices left. The final 14-factor model had reduced the original chi-square value from 6588.92 to 3527.21, but it also added an additional 13 indices to the single-index model, so it was important to find that perfect balance between the goodness of fit and the simplicity of the model. To do this, the Akaike information criterion (AIC) was used. The AIC allows for the model selection by measuring the quality of the models relative to each other and using it, it was found that the perfect balance between the goodness of fit and the simplicity of the model is the seven-factor model. This was found by first measuring the AIC value of the single-index model which was 191.55 and the idea here was to find which model reduces the chi-square the most without increasing the complexity of the model (AIC value) over the single-index model value. The seven-factor model reduces the chi-square value from 6588.92 to 4678.48 but it has an AIC value of 191.22, which is why this model was selected as the most optimum multi-index model built from the selected indices.

Model	Chi-square Sum	K	N	AIC
1 Factor Model	6588.92	2	36	191.55
2 Factor Model	5802.30	3	36	188.97
3 Factor Model	5520.95	4	36	189.18
4 Factor Model	5289.31	5	36	189.64
5 Factor Model	5062.22	6	36	190.06
6 Factor Model	4862.34	7	36	190.61
7 Factor Model	4678.48	8	36	191.22
8 Factor Model	4515.11	9	36	191.94
9 Factor Model	4356.08	10	36	192.65
10 Factor Model	4157.44	11	36	192.97
11 Factor Model	3995.73	12	36	193.54
12 Factor Model	3840.25	13	36	194.11
13 Factor Model	3688.02	14	36	194.66
14 Factor Model	3527.21	15	36	195.05

The seven-factor model consists of the 10- and 30- Year U.S. Treasury Bond, Gold, Oil, CMA, RMW and a market factor. It was not surprising to find that both 10- and 30- Year U.S. Treasury Bonds provide significant explanation to the stock prices. This confirmed the findings of (Longstaff, 2004; Shen, 2012; Pilotte and Sterbenz, 2006). During an economic expansion, investors move the money from the bond market to the stock market, so the prices of bonds go down, but the yields go up to attract new investors. The principle is reversed in the time of economic contraction, but it is clear that there is significant correlation between the stock market and the bond market which our model confirmed as well.

Furthermore, in case of oil, findings in this report validate the findings of (P. Sadorsky, 1999) who found that the oil prices and oil price volatility play a significant role in the real stock returns. This is also confirmed from the fundamental standpoint because reduction in oil prices positively impact the prices of stocks due to the reduction in transportation costs and due to oil being used heavily in the industrial processes. This leads to lower manufacturing costs and a more thriving business.

The next two indices are from the Fama and French Five Factor model, the CMA and RMW, which were introduced in 2015 as the addition to the original three factors firstly published in 1993. That is 22 years after the original paper was published and the two factors that they introduced have been validated in this paper as indices that explain a significant price movement of stocks which was their original finding as well.

Even though the indices selected by the AIC criterion have been confirmed in the literature to carry influence over the stock price movements, it is another thing to have them working well in a multi-index model. To validate the findings and conclusions provided by the AIC criterion, the top seven performing multi-index models had to be tested on different years. It was decided that the models will be tested on the year 2019 for a total of 12 trading months and years 2020-2021 for a total of 24 trading months to compare it to the performance of the model in 2015-2018. The testing process required the extraction of the betas for each company and each index from the years 2015-2018 and then applying those betas to the multi-index models in the year 2019 and 2020-2021. To be able to compare the results of the chi-square, due to the different testing periods, a normalised chi-square value has been created by dividing the chi-square test value by the number of testing months. The results can be seen in the Table 33. and comparing the findings between the year 2019 and 2015-2018, it can be seen that the penalty of the AIC was more lenient than it should have been. The results of the 2019 chi-square test indicate that the most predictive model is actually the three-factor model, because starting with the single-index model and adding the first two indices, the chi-square reduces in value in the testing period as it should do, but with the addition of the third extra index, the value begins to increase. This is pointing out that the predictiveness of the four-factor model is beginning to decrease and that is the case with the further addition of indices. Nevertheless, it should be pointed out that increase in the chi-square value is not drastic because the value of chi-square for the single-index model is 156.14 and the seven-factor model is 159.53. This implies, that our model is actually working well in the year 2019, but not as well as the three-factor model, which according to the test is the preferred one.

The year 2020-2021 was selected to test the performance of our models in the time of uncertainty and extreme market events caused by the COVID-19 pandemic. The chi-square of the single-index model was 162.24 but values immediately increased with the addition of extra indices, ending up with a value of 202.68 by the time the seven-factor model was tested. Comparing this to the 2019 results, it can be seen that the jumps in chi-square value are much higher and this implies that our multi-index model is not good at predicting share prices in the time of market uncertainty or extreme events.

Table 33 - Normalised Chi-square values

Model	Chi-square Normalised		
	2015-2018	2019	2020-2021
1 Factor Model	183.03	156.14	162.24
2 Factor Model	161.18	155.98	172.77
3 Factor Model	153.36	155.96	176.09
4 Factor Model	146.93	157.41	186.62
5 Factor Model	140.62	158.25	193.66
6 Factor Model	135.07	160.04	198.70
7 Factor Model	129.96	159.53	202.68

To summarize, the original thesis which focused on the expansion of the single-index model with the aim to increase the performance is validated by the findings in this report. During the first tests of multi-index models in 2015-2018, the seven-factor model provided significant improvement over the single-index model and was identified as the best balance between the goodness of fit and the simplicity of the model. The further validation of the seven-factor model by testing it in the year 2019, found the performance of the three-index model to be the best, hence, a more penalty focused information criterion is recommended for further research. Our conclusion is that the three-factor model which consists of the 10-Year U.S. Treasury Bond, Gold and the market factor will create the most predictive model for future use. However, it has been shown that this model works only in the normal market conditions. In the case of unusual market activity or extreme events the simple SIM is superior.

9. Conclusion

Building a quality multi-index model can be considered a form of art. Having countless possible inputs and finding the balance between the simplicity and goodness of fit of the model, the “perfect” model just does not exist. Nevertheless, with the right approach it is possible to narrow down the selection to several good models. To do this, in this paper the starting point was a single-index model which was expanded to a multi-index model with a list of carefully selected indices. The initial evaluation of the model quality was done by correlation coefficients which was replaced by a more advanced statistical method known as the chi-square test. Using this method, it was possible to simulate 91 multi-index models using the data for 2015-2018 and narrow it down to 14 good ones. The next part of the report was focusing on finding that balance between simplicity and quality. This was done using the Akaike information criterion, which indicated a cut-off point at a multi-index model with 7 indices, which meant that all the way from single-index model to the seven-factor model, there is a good trade-off between simplicity and quality. It is up to the reader to decide, based on the personal preferences, what is of more importance, the fit quality or simplicity. In case of maximizing the quality of the model, the seven-factor model with respect to the single index model had a reduction in chi-square from 6588.92 to 4678.48. The model consists of the market factor, 10- and 30- Year U.S. Treasury Bond, Gold, Oil, CMA and RMW which were proven to provide the most explanation to the stock price movements and these findings were also confirmed by cited literature. Expanding on these findings, it was important to test the seven-factor model in a different year to test the predictiveness of the model. Using the extracted betas from the original seven-factor model and applying it to the year 2019, it was actually found that seven-factor model was overfitted and the top-performing model was the three-factor model. This test was carried out by observing the normalised values of the chi-square test and it was seen that the values decreased from the single-index model to three-factor model but started to increase afterwards. This points to the conclusion that the Akaike information criterion was over lenient in the penalization process. However, the increase in the standardized chi-square value was not drastic, but if the seven-factor model is assumed for prediction of the future stock-price movements, the results should be taken with some care.

Furthermore, the next objective was to test the performance of our multi-index models in the time of extreme market events, so the two years 2020-2021 of pandemic were selected, which had unusual market movements. The results showed that none of the multi-index models were performed better than the SIM. This indicates a weakness in our models as their performance is questionable during extreme market events. One could speculate that the seven-factor model is better for short-term predictions, but the simple three-index model would be better for long term analysis.

However, this might come from the fact that during the period of extreme market events, everything is much more correlated to the overall market movement, hence the single-index model had much more explanatory power.

Referencing back to the original thesis of this report which was to improve upon the single-index model performance by the careful selection of indices and creation of the multi-index models, it can be concluded that the thesis has been successfully proven. Based on the historical data, the top performing model was a seven-factor model but in case of predictiveness, the overall best model is a three-factor model, consisting of the 10-Year U.S. Treasury Bond index, Gold index and the market index.

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APPENDIX A

APPENDIX B

Date:	Returns									
	KO	WMT	TSN	KMB	ADBE	AAPL	MCHP	MSFT	JPM	MCO
30/04/2016	-0.03427	-0.02365	-0.0126	-0.06929	0.004478	-0.13992	0.008091	-0.09705	0.075108	-0.0087
31/05/2016	-0.00446	0.066221	-0.02876	0.014777	0.055721	0.071773	0.071575	0.070197	0.032753	0.034591
30/06/2016	0.024162	0.031647	0.047194	0.08988	-0.037	-0.04266	-0.0178	-0.03453	-0.04795	-0.04998
31/07/2016	-0.0375	-0.00068	0.101961	-0.05768	0.02161	0.090063	0.096139	0.10768	0.037464	0.131256
31/08/2016	-0.00458	-0.01427	0.028796	-0.0115	0.045473	0.023652	0.1193	0.02008	0.055182	0.028936
30/09/2016	-0.0176	0.009519	-0.01191	-0.00793	0.060893	0.065504	0.003715	0.002437	-0.01348	-0.00377
31/10/2016	0.001891	-0.02912	-0.05116	-0.09299	-0.00949	0.004334	-0.02559	0.040278	0.047657	-0.07167
30/11/2016	-0.0403	0.005856	-0.19506	0.010489	-0.04372	-0.0216	0.099024	0.012468	0.157522	0.003564
31/12/2016	0.027509	-0.01158	0.085724	-0.00488	0.001362	0.047955	-0.03067	0.031198	0.076338	-0.06199
31/01/2017	0.002653	-0.03443	0.017996	0.061427	0.101311	0.047747	0.049883	0.040393	-0.01381	0.099713
28/02/2017	0.009382	0.062781	-7.35E-05	0.094279	0.043747	0.133778	0.082197	-0.00436	0.070779	0.081766
31/03/2017	0.020367	0.023685	-0.01359	0.000334	0.099628	0.04869	0.017375	0.029384	-0.03068	0.006016
30/04/2017	0.01673	0.043008	0.041322	-0.01428	0.027741	-6.97E-05	0.024397	0.039478	0.001865	0.056051
31/05/2017	0.053766	0.052487	-0.10423	-0.00015	0.060715	0.067807	0.107195	0.026005	-0.05575	0.004478
30/06/2017	-0.00552	-0.03715	0.092257	0.002647	-0.00296	-0.05721	-0.07347	-0.01303	0.112599	0.027269
31/07/2017	0.022074	0.05695	0.011656	-0.04608	0.035704	0.032704	0.037056	0.054693	0.009901	0.081772
31/08/2017	-0.00633	-0.01786	0.002681	0.001055	0.059185	0.106999	0.089143	0.033953	-0.00991	0.021213
30/09/2017	-0.00399	0.000897	0.112954	-0.03774	-0.03854	-0.06024	0.034332	-0.00375	0.050831	0.038648
31/10/2017	0.021551	0.117353	0.034918	-0.04393	0.174152	0.096808	0.055914	0.11666	0.059555	0.022987
30/11/2017	0.003594	0.113618	0.135393	0.064439	0.036024	0.020278	-0.07872	0.016984	0.038863	0.068844
31/12/2017	0.002403	0.026336	-0.01435	0.015579	-0.03433	-0.01525	0.010231	0.016277	0.023154	-0.02773
31/01/2018	0.037271	0.079494	-0.06118	-0.03033	0.139922	-0.01064	0.083523	0.110708	0.087269	0.096064

28/02/2018	-0.09183	-0.15563	-0.01878	-0.05197	0.046906	0.068185	-0.06198	-0.00841	-0.00147	0.034186
31/03/2018	0.013729	-0.00566	-0.016	0.001842	0.033233	-0.05805	0.027325	-0.02666	-0.04788	-0.03344
30/04/2018	-0.00507	-0.00573	-0.04222	-0.05984	0.025546	-0.01502	-0.08428	0.024652	-0.00581	0.005579
31/05/2018	-0.00486	-0.06104	-0.03341	-0.02598	0.12491	0.135124	0.168552	0.061467	-0.01627	0.054307
30/06/2018	0.029084	0.037679	0.020454	0.054993	-0.02194	-0.00942	-0.06603	-0.00233	-0.02626	-5.85E-05
31/07/2018	0.063156	0.041798	-0.16267	0.080881	0.003568	0.027983	0.027268	0.075753	0.109162	0.003283
31/08/2018	-0.04418	0.080542	0.094713	0.014755	0.076958	0.200422	-0.07515	0.062993	-0.00322	0.042997
30/09/2018	0.045163	-0.02034	-0.05222	-0.00794	0.024439	-0.0083	-0.08276	0.018161	-0.01519	-0.06078
31/10/2018	0.036588	0.067831	0.006551	-0.08219	-0.08961	-0.03048	-0.16639	-0.0661	-0.02709	-0.1299
30/11/2018	0.060955	-0.02623	-0.00967	0.106136	0.020874	-0.1812	0.145825	0.042684	0.019905	0.096671
31/12/2018	-0.06052	-0.04088	-0.09415	-0.00354	-0.09825	-0.1167	-0.04107	-0.08405	-0.12204	-0.11963
31/01/2019	0.016473	0.028771	0.15955	-0.02247	0.095385	0.055154	0.117492	0.028158	0.068844	0.131891
28/02/2019	-0.05797	0.032975	0.001895	0.048932	0.059237	0.044777	0.085244	0.077358	0.008309	0.095381
31/03/2019	0.042547	-0.00945	0.126014	0.070005	0.0152	0.097026	-0.04501	0.052754	-0.02999	0.046038
30/04/2019	0.046948	0.054445	0.080369	0.036158	0.085407	0.056436	0.204074	0.107343	0.155171	0.085758
31/05/2019	0.001427	-0.00837	0.01673	-0.00382	-0.06344	-0.12421	-0.19528	-0.04948	-0.08695	-0.06742
30/06/2019	0.044525	0.089215	0.063908	0.050249	0.087671	0.130519	0.083344	0.083118	0.055115	0.067968
31/07/2019	0.033582	-0.001	-0.01536	0.017782	0.014288	0.076395	0.089043	0.017244	0.044913	0.097435
31/08/2019	0.045791	0.040247	0.17518	0.040251	-0.04802	-0.01646	-0.08195	0.015037	-0.05293	0.008138
30/09/2019	-0.00367	0.038684	-0.07416	0.013981	-0.02903	0.072962	0.076219	0.008487	0.071272	-0.04987
31/10/2019	-0.00018	-0.01196	-0.03889	-0.06455	0.006081	0.110685	0.014853	0.031216	0.069935	0.07743
30/11/2019	-0.01159	0.015606	0.095502	0.026039	0.113698	0.077554	0.006639	0.059462	0.054755	0.029396
31/12/2019	0.036517	0.002377	0.012793	0.01649	0.065519	0.098784	0.107679	0.041749	0.057984	0.047382
31/01/2020	0.055104	-0.0366	-0.09238	0.041367	0.064674	0.05401	-0.06914	0.079455	-0.04441	0.081631
29/02/2020	-0.08408	-0.05948	-0.17423	-0.08412	-0.01714	-0.1147	-0.06633	-0.04569	-0.12277	-0.06334
31/03/2020	-0.16549	0.059832	-0.14684	-0.01822	-0.07789	-0.06976	-0.25256	-0.02654	-0.22461	-0.11886

Table 34 - Discrete returns of securities

Date:	Returns									
	ETFC	TRV	CVX	DVN	MPC	OKE	ATVI	FB	NFLX	OMC
30/04/2016	0.028175	-0.05835	0.071069	0.263848	0.051102	0.231074	0.018617	0.0305	-0.11934	-0.00312
31/05/2016	0.107625	0.038581	-0.0011	0.040658	-0.10064	0.196404	0.138961	0.010461	0.139287	0.004339
30/06/2016	-0.15776	0.049064	0.037921	0.006152	0.089865	0.09711	0.009424	-0.03813	-0.10812	-0.01574
31/07/2016	0.067688	-0.02369	-0.02242	0.056	0.037671	-0.05606	0.013374	0.084529	-0.00251	0.009817
31/08/2016	0.051834	0.021425	-0.0082	0.131923	0.088603	0.061119	0.03013	0.017589	0.067945	0.046664
30/09/2016	0.103867	-0.02959	0.023265	0.019418	-0.04517	0.095969	0.070824	0.017047	0.011288	-0.00662
31/10/2016	-0.03297	-0.05561	0.017781	-0.14101	0.07391	-0.04568	-0.02551	0.021205	0.267073	-0.06094
30/11/2016	0.225497	0.047791	0.075674	0.275535	0.087594	0.134214	-0.15196	-0.09596	-0.06303	0.089201
31/12/2016	0.004057	0.086247	0.055038	-0.05386	0.070821	0.045148	-0.01366	-0.02846	0.05812	-0.01488
31/01/2017	0.080808	-0.0379	-0.05395	-0.00285	-0.04568	-0.02932	0.113542	0.132725	0.136591	0.006345
28/02/2017	-0.0785	0.037867	0.019999	-0.04787	0.039702	-0.01923	0.122358	0.040055	0.010092	-0.00642
31/03/2017	0.011011	-0.00847	-0.0456	-0.03639	0.018952	0.025717	0.111572	0.04803	0.039963	0.01968
30/04/2017	-0.00975	0.009292	-0.00624	-0.05345	0.007914	-0.04022	0.047934	0.057726	0.0297	-0.04744
31/05/2017	0.001737	0.026221	-0.02031	-0.13953	0.028625	-0.05569	0.121148	0.008053	0.071419	0.019484
30/06/2017	0.098815	0.019374	0.008214	-0.05742	0.005573	0.04992	-0.01724	-0.00317	-0.08377	-0.00307
31/07/2017	0.078096	0.012329	0.046583	0.041914	0.069941	0.084548	0.073128	0.121009	0.215849	-0.05018
31/08/2017	0.000244	-0.05395	-0.00448	-0.05734	-0.05599	-0.02954	0.061185	0.016071	-0.03826	-0.08077
30/09/2017	0.063399	0.017279	0.091805	0.173393	0.069209	0.02308	-0.01602	-0.0064	0.038006	0.030975
31/10/2017	-0.00046	0.081048	-0.0137	0.005176	0.065264	-0.02057	0.015191	0.053784	0.083154	-0.09289
30/11/2017	0.104382	0.023556	0.036361	0.044173	0.055089	-0.03013	-0.04718	-0.01599	-0.04505	0.063254
31/12/2017	0.029705	0.005913	0.052105	0.076182	0.053489	0.029865	0.014744	-0.00406	0.02335	0.027744
31/01/2018	0.063143	0.105279	0.001278	-0.00072	0.049863	0.115393	0.17072	0.059107	0.408106	0.052451
28/02/2018	-0.00892	-0.07284	-0.09825	-0.25864	-0.06891	-0.04298	-0.01349	-0.04586	0.077987	-0.00548
31/03/2018	0.060885	0.004199	0.018942	0.038418	0.141274	0.010474	-0.07278	-0.10391	0.013625	-0.03892
30/04/2018	0.095109	-0.05228	0.097071	0.142812	0.024621	0.072152	-0.01645	0.076413	0.057931	0.013623
31/05/2018	0.044001	-0.0234	0.002189	0.144233	0.061231	0.13185	0.068727	0.115	0.125264	-0.02145
30/06/2018	-0.03457	-0.04236	0.017136	0.059484	-0.11224	0.024501	0.076294	0.013244	0.113282	0.066597

31/07/2018	-0.02207	0.063757	-0.00127	0.023885	0.152081	0.008736	-0.038	-0.11188	-0.1379	-0.09755
31/08/2018	-0.01588	0.011219	-0.05286	-0.04621	0.023942	-0.0529	-0.01798	0.018252	0.089584	0.007119
30/09/2018	-0.10992	-0.0086	0.032247	-0.06787	-0.02819	0.028524	0.153814	-0.06413	0.017542	-0.01014
31/10/2018	-0.0539	-0.03531	-0.08693	-0.18878	-0.11904	-0.03231	-0.16997	-0.07704	-0.19338	0.09262
30/11/2018	0.058074	0.041877	0.075705	-0.16574	-0.06833	-0.05105	-0.27762	-0.07365	-0.05186	0.035657
31/12/2018	-0.16083	-0.07579	-0.08534	-0.16366	-0.09438	-0.12176	-0.06636	-0.06771	-0.06455	-0.04067
31/01/2019	0.06657	0.048351	0.053865	0.182343	0.122861	0.206741	0.014387	0.271569	0.2684	0.063353
28/02/2019	0.049936	0.058707	0.053485	0.107317	-0.05647	0.000779	-0.10796	-0.03144	0.054786	-0.02799
31/03/2019	-0.05226	0.038055	0.030105	0.072393	-0.03483	0.086835	0.089125	0.032456	-0.0043	-0.02731
30/04/2019	0.091105	0.048046	-0.02533	0.018378	0.017043	-0.01514	0.058862	0.160238	0.039208	0.096452
31/05/2019	-0.11319	0.012661	-0.04239	-0.21717	-0.23657	-0.06345	-0.10039	-0.08237	-0.07357	-0.03336
30/06/2019	-0.00446	0.032819	0.093017	0.137563	0.215047	0.081578	0.08831	0.087508	0.07003	0.068032
31/07/2019	0.093946	-0.0194	-0.01069	-0.0533	0.009127	0.018457	0.032627	0.006373	-0.12069	-0.02111
31/08/2019	-0.14151	0.002319	-0.03395	-0.18556	-0.11755	0.030037	0.038162	-0.04407	-0.09053	-0.05186
30/09/2019	0.046718	0.017229	0.007475	0.098092	0.234505	0.03381	0.04585	-0.04088	-0.08895	0.037964
31/10/2019	-0.04349	-0.11857	-0.02074	-0.15711	0.052675	-0.05238	0.058768	0.076202	0.073948	-0.01418
30/11/2019	0.06349	0.043186	0.018462	0.079388	-0.04391	0.03098	-0.02142	0.052126	0.094812	0.029667
31/12/2019	0.024153	0.007771	0.028857	0.190963	-0.00643	0.065025	0.083713	0.017903	0.028316	0.027739
31/01/2020	-0.06061	-0.03892	-0.11095	-0.16365	-0.09544	0.001705	-0.01582	-0.01627	0.066508	-0.07048
29/02/2020	0.07697	-0.08973	-0.11862	-0.2523	-0.12117	-0.10886	-0.00598	-0.04675	0.069373	-0.08007
31/03/2020	-0.25033	-0.16529	-0.2237	-0.5699	-0.5019	-0.67311	0.023224	-0.13337	0.017532	-0.19966

Table 35 - Discrete returns of securities (10-20)